THE UNIVERSITY OF CHICAGO

FINITE GROUP ACTIONS ON THE MODULI SPACE OF SELF-DUAL CONNECTIONS

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CHAPTER 1

INTRODUCTION

In this paper we would like to study finite group actions of the moduli space of self-dual connections.

Let G be a finite group. Let M be a simply connected, closed and smooth 4-manifold with a positive definite intersection form, and a smooth action of G on it. Let $\pi: E \to M$ be a quaternion line bundle with instanton number one and with a G-action on E through bundle isomorphism such that π is a G-map. The moduli space $\mathcal{M}$ which is the set of self-dual connections on E modulo the group $\mathcal{G}$ of gauge transformations, is a G space but may not be a manifold, when we start with G-invariant metric on M, because of nonvanishing cohomology groups and reducible connections. To make $\mathcal{M}$ a manifold, Donaldson used a compact perturbation of a Fredholm map, and Uhlenbeck found generic metrics on M. We cannot use these methods directly to make $\mathcal{M}$ a G-manifold, because the perturbation is not a G-equivariant and Uhlenbeck's method need not yield a G-invariant metric.

We can regard this G-action on this bundle as a subgroup of a generalized gauge group. From this G-action on the bundle, we can define naturally a G-action on the set C of all connection, $\mathcal{G}$, $\Omega^n(\mathcal{G}_E)$, and $\Omega^n(E)$ where $\mathcal{G}_E$ is the associated Lie algebra bundle of E, then G

acts on $\mathcal{C}/\mathcal{G}$ and the moduli space $\mathcal{M}$.

There are two methods to transform this mysterious G-moduli space into a smooth G-moduli space with some singularities.

(A) First Method: We can find a G-invariant metric on M such that the fixed point set $\mathcal{M}^G$ is a manifold. We will use the Uhlenbeck argument [14] which was used to find generic metrics on M.

THEOREM 5.6: <u>There exists a Baire set in the</u> G-<u>invariant metrics which are obtained by averaging, such that</u> $\hat{\mathcal{M}}^G$ <u>is a smooth manifold in the moduli space</u> $\hat{\mathcal{M}}$ <u>of irreducible self-dual connections.</u>

To see the local G-structure of $\mathcal{M}$ at each self-dual G-invariant connection $\nabla \in \mathcal{M}^G$, we will use the Atiyah-Singer G-index Theorem [1], [3] for a G-invariant elliptic complex:

$$0 \longrightarrow \Omega^0(\mathcal{G}_E) \underset{\delta^\nabla}{\overset{\nabla}{\rightleftarrows}} \Omega^1(\mathcal{G}_E) \xrightarrow{d^\nabla} \Omega^2(\mathcal{G}_E) \longrightarrow 0$$

where δ^∇ is the formal adjoint of ∇.

From now without mention about G, we will assume $G = Z_2 \equiv \langle h \rangle$. Suppose that the G-fixed point set $F \equiv \{P_i\}_{i=1}^{n_1} \cup \{T^{\lambda_i}\}_{i=1}^{n_2}$ on M, where P_i is an isolated fixed point and T^{λ_i} is a Riemann Surface with genus λ_i.

THEOREM 4.13. <u>If a connection</u> ∇ <u>is irreducible (reducible)</u>, G-<u>invariant, in</u> $\mathcal{M}$, $h(\nabla) = g(\nabla)$, $(hg)^2 = +1$ <u>for some gauge transformation</u> g, <u>then we get</u>

$$\begin{cases} \dim H^1_{\nabla_+} - \dim H^2_{\nabla_+} = \frac{1}{4}(10 + 3A)\ (+1) \\ \dim H^1_{\nabla_-} - \dim H^2_{\nabla_-} = \frac{1}{4}(10 - 3A) \end{cases}$$

where $A = n_1 + \sum_{i=1}^{n_2} \chi(T^{\lambda_i}) - \mathrm{sign}(h:M)$, sign = signature, $H^*_{\nabla\pm}$ means the ± 1 eigenspace of hg.

THEOREM 4.13'. *If* ∇ *be a self-dual irreducible connection and* $h(\nabla) = g(\nabla), (hg)^2 = -1$ *for some gauge transformation* g, *then we have*

$$\begin{cases} \dim H^1_{\nabla_+} - \dim H^2_{\nabla_+} = \frac{1}{4}(10 + A) \\ \dim H^1_{\nabla_-} - \dim H^2_{\nabla_-} = \frac{1}{4}(10 - A) \end{cases}$$

where $H^*_\pm$ *is the* ± 1 *eigenspace of* hg.

THEOREM 4.13". *Let* ∇ *be a self-dual reducible connection and* $g(\nabla) = h(\nabla)$ *for some gauge transformation* $g \notin \Gamma_\nabla$. *Then we get*

$$\begin{cases} \dim H^1_{\nabla_+} - \dim H^2_{\nabla_+} = \frac{1}{4}(14 + A) \\ \dim H^2_{\nabla_-} - \dim H^2_{\nabla_-} = \frac{1}{4}(10 - A) \end{cases}$$

where $H^*_\pm$ *is the* ± 1 *eigenspace of* g_1hgg_2, *for some* $g_1, g_2 \in \Gamma_\nabla$.

If we consider the end part of the moduli space [19], [30] and Theorem 4.13 and Theorem 4.13', then we have as by product, THEOREM 6.6. The value $A = n_1 + \sum_{i=1}^{n_2} (T^{\lambda_i}) - \text{sign}(h:M) = 2.$

Also we can get this value of A from the Lefschetz fixed point Theorem.

Using these index calculation, we want to perturb the map $\psi: C/\mathcal{G} \to C \times_{\mathcal{G}} \Omega^2_-(\mathcal{G}_E)$ given by $\psi(\nabla) = (\nabla, R^{\nabla}_-)$ to one transverse to the zero section. This Fredholm G-map ψ is locally equivalent to the sum of a linear map and a nonlinear map with finite dimensional range G-equivariantly.

If we use Theorem 4.13 and Theorem 4.13", then we get

THEOREM 6.7. Suppose that ∇ is G-invariant, reducible self-dual in M, then there is a G-equivariant perturbation around ∇ in β such that the perturbed moduli space has a neighborhood at ∇ which is an open cone on Cp^2.

If we use Theorem 4.13 and Theorem 4.13', then we obtain

THEOREM 6.10. If ∇ is G-invariant, irreducible in M, then there is a G-invariant smooth compact perturbation around ∇ such that the perturbed new moduli space has a smooth 5-dimensional neighborhood at ∇.

We wish to apply a G-transversality technique of Petrie [22] to investigate G-transversality on a neighborhood of the fixed point set M^G. Consider a fiber bundle $F \to V \to X$ where $X = M^G$. {End of $M \cup$ nbd of reducible connections in $M^G\} \cap X \equiv X_0$, $F = \mathrm{Hom}^s_G(H^1_{\nabla_-}, H^2_{\nabla_-})$ = the surjective G-homomorphisms.

THEOREM 6.16. (i) To perturb ψ G-transversally throughout a neighborhood of M^G there are the obstruction cohomology classes $O_3(\psi) \in H^3(X,X_0;Z)$

(ii) If $O_3(\psi) = 0$, then the G-section ψ has a smooth compact G-perturbation $R_- + \sigma$ of the self-dual Yang-Mills equations which is transversal to the zero section throughout a small neighborhood of M^G.

Let $N(M^G)$ be a neighborhood of M^G such that ψ is transverse to the zero section throughout $N(M^G)$. For each point $\nabla \in M\backslash N(M^G)$, we can choose a local coordinate chart $O_{\nabla\cdot\varepsilon}$ in $C/\mathcal{G}$ such that $h(O_{\nabla\cdot\varepsilon}) \cap O_{\nabla\cdot\varepsilon} = \emptyset$. Let $K = M \setminus \{N(M^G) \cup$ End of $M \cup$ neighborhood of reducible self-dual connections$\}$. The compactness of K and the local splitting of ψ give us a G-map

$$\psi_1 : C/\mathcal{G} \times D^n(\eta) \longrightarrow C \times \Omega^2(\mathcal{G}_E) \quad \text{via} \quad \psi_1(x,w) = (x) + \sigma(x,w)$$

where σ is defined G-equivariantly for each ω in a η-ball $D^n(\eta) \subset R^n$ for some n.

THEOREM 8.6. For almost all $\omega \in D^n(\eta)$ the restriction map $\psi(\ ) + \sigma(\ ,\omega)$ is transversal to the zero section throughout a neighborhood of K.

Thus if the obstruction cohomology classes $O_3(\lambda) = 0$, then we have a smooth G-manifold M of dimension 5 with λ-singular points each of which has a cone neighborhood on Cp^2, where $\lambda = \text{rank } H^2(M;Z)$.

(B) Second Method: We would like to find a G-invariant metric on M such that a neighborhood of M^G is a G-manifold by modifying the Uhlenbeck generic argument [14] and by using a property which a finite group action on M is almost free. Throughout, the hat ^ stands for irreducible.

THEOREM 7.3. <u>There is an open</u> G-<u>set</u> O <u>of</u> $C^{\wedge} \times C^G$ <u>such that</u>

(i) <u>the restriction map</u> $\phi: O \to \Omega_-^2(\mathcal{G}_E)$ <u>is smooth and has zero as a regular value</u>

(ii) <u>if</u> $\phi(\nabla,\phi) = 0$ <u>and</u> $\nabla \in M^{\wedge G}$, <u>then</u> $\pi^{-1}(M^{\wedge G}) \times \{\phi\} \in O$, <u>where</u> $\pi: C^{\wedge} \to B^{\wedge}$ <u>is the projection and</u> $C = C^k(GL(TM))$.

Using Sard-Smale Theorem [27] for a Fredholm map, we have

THEOREM 7.6. <u>There is an open dense set in</u> C^G <u>such that the moduli space</u> $M^{\wedge}$ <u>of irreducible connections is a manifold in a</u> G-<u>neighborhood of</u> $M^{\wedge G}$ <u>for each metric in the dense set</u>.

Similar result Theorem 7.7 is obtained for the reducible connection. By Theorem 7.6 and Theorem 7.7, we get a following result.

THEOREM 7.8. There is a dense set in the C^G of the C^∞, G-invariant metrics on M such that the moduli space $\mathcal{M}$ is a manifold in a G-neighborhood of the fixed point set $\mathcal{M}^G$ for each metric in the dense set. Moreover for these metrics Petrie obstruction classes vanish.

This result is true for the cyclic group G of order 2^n (Theorem 7.9).

Again by the perturbation of the free part in $\mathcal{M}$ (Theorem 8.6) we have a smooth G-manifold $\mathcal{M}$ of dimension 5 with λ-singular points, each of which has a cone neighborhood of Cp^2, where $\lambda = \text{rank}\, H^2(M;Z)$.

In Chapter IX for an odd prime p, we have

THEOREM 9.7. Let a Z_p-action on $E \to M$ have the fixed point set $F = p_i\{\Sigma_i\}_{i=1}^{k_1} \cup \{T^{\lambda_i}\}_{i=1}^{k_2}$ on M and let D be the induced operator by a Z_p-invariant fundamental elliptic complex. Then we have

$$\text{Index}_{g^n}(D) = \sum_{i=1}^{k_1}\left(-\frac{3}{2}\right)\left[1+\cot \frac{n\pi s_i}{p}\right] + \sum_{j=1}^{k_2}(-6)\left[\frac{(1-\lambda_i)+(m_{T^{\lambda_i}}+\lambda_j-1)e^{\frac{2n\pi i t_j}{p}}}{\left(1-e^{\frac{2n\pi i t_j}{p}}\right)^2}\right]$$

where $g^n \in Z_p$, $m_{T^{\lambda_j}}$ = self-intersection number of T^{λ_j} and r_i, s_i, t_i are determined by representations of the normal bundles.

CHAPTER II

CONSTRUCTION OF Z_n ACTIONS ON SU(2)-BUNDLE WITH $k = 1$

Recall that HP^n is the set of 1-dimensional quaternion subspaces in the (n+1)-dimensional quaternion space H^{n+1}, $E = \{(\ell,v) \in HP^n \times H^{n+1}: v \in \ell\}$ The projection $p: E \to HP^n$ given by $p(\ell,v) = \ell$ is a natural quaternion line bundle. The associated unit sphere bundle of $E \to HP^n$ is just the Hopf bundle $S^{4n+3} \to HP^n$ which is 4n-dimensional classifying of SU(2)-bundles. In case $n = 1$, $HP^1 = S^4$ and the Hopf-bundle $S^7 \to S^4$ is 4-dimensional classifying of SU(2)-bundles with $c_2(E)[S^4] = 1$. There is a well-known fact;

THEOREM 2.1. _Let_ M _be a compact oriented_ 4-_manifold. Then there are natural_ 1-1 _correspondences_

$$\{\text{equivalence classes of SU(2)-bundles on } M\}$$

$$\longleftrightarrow [M^4,S^4] \longleftrightarrow H^4(M;Z) = Z.$$

We would like to give a finite group action on this 4-dimensional classifying bundle $S^7 \to S^4$ of SU(2)-bundles. $S^7 = \{(x,y) \in H^2: |x|^2 + |y|^2 = 1\}$, $S^4 = H \cup \{\infty\}$.

The projection $\pi: S^7 \to S^4$ is given by $\pi(x,y) = \begin{cases} xy^{-1} & \text{if } y \neq 0 \\ \infty & \text{if } y = 0 \end{cases}$

First the unit quaternion SU(2) acts on S^7 by $a(x,y) = (xa,ya)$ for $a \in SU(2)$ and $(x,y) \in S^7$ and $S^4 = S^7/SU(2)$.

Secondly a finite cyclic group Z_p imbeds in SU(2) because $S^1 \subset SU(2)$. Define Z_p action on S^7 by $b(x,y) = (bx,y)$ for $b \in Z_p \subset SU(2)$ and $(x,y) \in S^7$. Clearly the action of SU(2) and the action of Z_p on S^7 are commutative, and the projection $\pi: S^7 \to S^4$ is a Z_p-map. Extend these actions to the acsociated SU(2)-vector bundle of the Hopf-bundle $S^7 \to S^4$. Give a natural SU(2) action on C^2. Then the group actions on the associated vector bundle $S^7 \times_{SU(2)} C^2 \to S^4$ are defined by $a \in SU(2)$,

$$(x,y,v) \in S^7 \times_{SU(2)} C^2 \text{ and } g \in Z_p \subset SU(2) \text{ via}$$

$$a(x,y,v) = (xa,ya,av) \text{ and}$$

$$g(x,y,v) = (gx,y,v) \text{ respectively.}$$

More precisely, let $S^4 = D^4_+ \cup D^4_-$, $E = S^7 \times_{SU(2)} C^2$. Then we have

$$E|_{D^4_+} = D^4_+ \times C^2, \quad E|_{D^4_-} = D^4_- \times C^2, \quad E|_{D^4_+ \cap D^4_-} = E|_{S^3} = S^3 \times C^2$$

The transition function for the bundle $E \to S^4$ is just the identity function $D^4_+ \cap D^4_- = S^3 \xrightarrow{\tau} SU(2) = S^3$, $\tau(a) = a$. So the bund $E \to S^4$ is obtained by identifying $S^3 \times C^2 \to S^3 \times C^2$ via $(a,v) \mapsto (a,av)$. Finally define Z_p action on this bundle. For $g \in Z_p \subset SU(2)$ define a Z_p-action

$$\text{on } D^4_+ \times C^2 \text{ via } g(a,v) = (ag^{-1},gv)$$

$$\text{on } D^4_- \times C^2 \text{ via } g(a,v) = (ag^{-1},v)$$

and on the overlaping part $(D^4_+ \cap D^4_-) \times C^2$ for $(a,v) \in S^3 \times C^2$, $g \in Z_p$,

$$\begin{array}{ccc} (a,v) & \xrightarrow{\text{transition map}} & (a,av) \\ Z_p\text{- action on } D^4_+ \times C^2 \Big\downarrow & & \Big\downarrow Z_p\text{-action on } D^4_- \times C^2 \\ (ag^{-1},gv) & \xrightarrow{\text{transition map}} & (ag^{-1},av) \end{array} \qquad \text{commutes.}$$

Thus Z_p-action is a well-defined smooth action.

THEOREM 2.2. <u>On</u> 4-<u>classifying</u> SU(2)-<u>bundle</u> $E \to S^4$ <u>we may give a nontrivial</u> Z_p <u>action</u>.

Let M be a simply connected oriented smooth 4-manifold and let $E_1 \to M$ be a quaternion line bundle with Euler class -1. Since the classifying SU(2)-bundle $E \to S^4$ has Euler class -1. By Theorem 2.1 $E_1 \to M$ can be constructed by a degree one map $M \to S^4$ as follows.

Let $I(M)$ be the injective radius of M, $I(M) > 0$ since M is a compact smooth manifold. For any point $x \in M$, and $t \in (0, \frac{I(M)^2}{4})$ let V be a neighborhood of x such that V is diffoemorphic with an open set of T_xM by the exponential map and V contains $B_x(I(M))$.

Choose a smooth cutoff function $\lambda: [0,\infty] \to [0,1]$ which is an approximation of the characteristic function $\chi_{[0,1]}$. We may define a smooth map $\phi_{t,x}: M \to S^4$ by $\phi_{t,x}(y) = \exp^{-1}(y)$ on the $\sqrt{t}$-neighborhood $B_{\sqrt{t}}(x)$ of x and $\phi_{t,s}(M\backslash V) = \infty$ (cf. F.U)

Here we identify $S^4 = R^4 \cup \{\infty\}$. The ball $B_{\sqrt{t}}(x) \subseteq M$ is radial isometrically mapped onto $B_{\sqrt{t}}(0) \subset T_x(m)$ by $\phi_{t,x}$ and the annulus $B_{2\sqrt{t}}(x) - B_{\sqrt{t}}(x)$ in M is stretched onto the exterior of $B_{\sqrt{t}}(0)$ in T_xM, and outsides to ∞. Let $\Phi_{t,s} = T_t \circ \phi_{t,x}$, where $T_t: S^4 \to S^4$ is a dilation with scale factor t. Then the induced bundle $\Phi^*_{t,x}E \to M$ is a SU(2)-vector bundle with instanton number 1 because $\Phi_{t,x}$ is a degree one map M to S^4. Note the induced connection on $\Phi^*_{t,x}E$ has the half of the action on $B_{\sqrt{t}}(x)$.

By above construction and Theorem 2.2, we have following corollary.

COROLLARY 2.3. <u>Suppose a Z_p-action on M has an isolated fixed point which has the same isotropy representation as the standard Z_p-action on C^2. Then any SU(2)-bundle on M^4 with $k = 1$ has a non-trivial Z_p action.</u>

EXAMPLE 2.4. Let $M^4 = Cp^2$, The complex projective plan Cp^2 is obtained by adding a complex line at infinite of $C^2 \equiv R^4$. If $S^2 = Cp^1 \subset Cp^2$ is any projective line, then $Cp^2\backslash S^2 = R^4$. As above

$\Phi_{\lambda,0}$: $Cp^2 \supset R^4 \to S^4$ is defined. Then the fixed point set is $\{0\} \cup Cp^2 \setminus B_0(2\sqrt{t})$ which is homotopic to $\{p, S^2\}$.

CHAPTER III

FINITE GROUP ACTIONS ON CONNECTIONS

Let G be a finite group. In this chapter we would like to introduce G-actions on the set of all connections and various bundles. This group action is first introduced by Fintushel and Stern in [12].

Choose Riemannian metric on a vector bundle $E \to M$ with respect to which G acts by isometries. Let $\Omega^k(E) = \Gamma(\Lambda^k T^* M \otimes E)$ be the k-forms on M with values in E.

DEFINITION 3.1. A Riemannian Connection on E is a linear map

$\nabla: \Omega^0(E) \longrightarrow \Omega^1(E)$ such that

$\nabla(f\sigma) = df \otimes \sigma + f\nabla(\sigma)$ and

$d\langle\sigma_1,\sigma_2\rangle = \langle\nabla\sigma_1,\sigma_2\rangle + \langle\sigma_1,\nabla\sigma_2\rangle$

for any $f \in C^\infty(M)$ and any $\sigma, \sigma_1, \sigma_2 \in \Omega^0(E)$.

Given a metric on a compact manifold M there is a unique Riemannian connection ∇^M on $TM \to M$ which is torsion free. This is called a Levi-Civita connection. We shall always use this connection on TM.

We extend a Riemannian connection ∇ on E to the generalized de-Rham sequence

$$\Omega^0(E) \xrightarrow{\nabla} \Omega^1(E) \xrightarrow{d^\nabla} \Omega^2(E) \xrightarrow{d^\nabla} \cdots\cdots$$

for any $\phi \in \Omega^p(E)$, and any smooth vector fields $v_0, \ldots, v_p$

(3.2) $$(d^\nabla \phi)_{v_0 \cdots v_p} = \sum_{i=0}^{p} (-1)^i \, \nabla_{v_i} (\phi(v_0, \cdots, \hat{v}_i \cdots v_p))$$

$$+ \sum_{i<j} (-1)^{i+j} \phi([v_i, v_j], v_0, \cdots, v_i \cdots v_j \cdots v_p).$$

where $[v_i, v_j] = \nabla^M_{v_i} v_j - \nabla^M_{v_j} v_i$.

In particular on $\Omega^1(E) = \Gamma(T^*M \otimes E)$ for $\phi_1 \otimes \phi_2 \in \Gamma(T^*M \otimes E)$

$d^\nabla(\phi_1 \otimes \phi_2) = \tilde{\nabla}^M \phi_1 \otimes \phi_2 + \phi_1 \otimes \nabla \phi_2$ where $\tilde{\nabla}^M$ is the dual connection of ∇^M i.e. $(\tilde{\nabla}^M_v w)(x) = v[w(x)] - w(\nabla^M_v x)$ where w is an one form and v, x are vector fields.

The curvature of a connection ∇ is the 2-form $R^\nabla \in \Omega^2(\mathrm{Hom}(E,E))$ with values in $\mathrm{Hom}(E,E)$, defined for smooth vector fields v, w by

(3.3) $$R^\nabla_{v,w} = \nabla_v \nabla_w - \nabla_w \nabla_v - \nabla_{[v,w]}.$$

REMARK. (i) The curvature R^∇ is a zero-order tensorial differential operator on E.

i.e. $f(R^\nabla_{v,w}) = R^\nabla_{fv,w}\phi = R^\nabla_{c,fw}\phi = R^\nabla_{v,w}f\phi$ for $f \in C^\infty(M)$.

(ii) The associated Lie algebra bundle $\mathcal{G}_E$ of E is defined by

$$\mathcal{G}_E = \{\phi \in \mathrm{Hom}(E,E) \,|\, \phi_x \in Su(2) = Su(E_x) \quad \text{for} \quad x \in M\}$$

since $\langle R^\nabla_{c,q}\phi_1,\phi_2\rangle = -\langle\phi_1,R_{v,w}\phi_2\rangle$ for $\phi_1,\phi_2 \in \Omega^0(E)$, $\forall v,\forall w \in TM$. Thus $R^\nabla \in \Omega^2(\mathcal{G}_E)$.

(iii) Since $d^\nabla(R^\nabla) = d^\nabla R^\nabla - R^\nabla D^\nabla = d^\nabla d^\nabla d^\nabla - d^\nabla d^\nabla d^\nabla = 0$. We have the Bianchi identity for R^∇.

(iv) A connection ∇ on E induces a connection ∇ on $\mathcal{G}_E$ by $\nabla(\phi) = [\nabla,\phi]$ $\forall\phi \in \Omega^0(\mathcal{G}_E)$.

We can iefine a metric on $\Lambda^p T^* M \times E$ which is induced from the metrics on E and M. The pointwise inner product gives an L^2 norm in $\Omega^k(E)$ by setting $(\phi_1,\phi_2) = \int_M\langle\phi_1,\phi_2\rangle$ for $\phi_1,\phi_2 \in \Omega^k(E)$.

We have the formal adjoint $\delta^\nabla\colon \Omega^{k+1}(E) \to \Omega^k(E)$ of d^∇ with the property that

<u>(3.4)</u> $(d^\nabla\phi_1,\phi_2) = (\phi_1,\delta^\nabla\phi_2)$ for all $\phi_1 \in \Omega^k(E)$, $\phi_2 \in \Omega^{k+1}(E)$. More explicitly we can write δ^∇ in terms of the Riemannian Connection.

<u>(3.5)</u> $(\delta^\nabla\phi)_{v_1\cdots v_k} = -\sum_{i=1}^{4}(\nabla_{ei}\phi)(e_i,v_1\cdots v_k)$ where $\{e_1\cdots e_4\}$ is a orthonormal basis of T_xM since M is compact, by the partition of unity there are SU(2)-connections on our SU(2)-bundle $E \to M$. Let C be the space of all SU(2)-connections on E. By the gauge group $\mathcal{G}$

of a SU(2) bundle $E \to M$, we mean the group $\mathcal{G}$ of smooth bundle automorphisms preserving the metric and SU(2)-structure on E.

More formally let $p \to M$ be the associated principal bundle of E . and let $P \times_{SU(2)} SU(2) \to M$ be the associated the Lie group bundle, where SU(2) acts by adjoint. This lie group bundle is in general not a principal bundle. Then we have that the gauge group $\mathcal{G} = \Gamma(P \times_{SU(2)} SU(2))$ is the space of the sections.

There is a natural action of the gauge group $\mathcal{G}$ on the space C of the connections given by

(3.7) $g(\nabla) = g \circ \nabla \circ g^{-1}$ for all $\nabla \in C$, $g \in \mathcal{G}$.

LEMMA 3.7. <u>The space</u> C <u>of</u> SU(2)-<u>connections on</u> E <u>is an affine space having</u> $\Omega^1(\mathcal{G}_E)$ <u>as the vector group of translations</u>.

Proof. Let ∇_1 and ∇_2 be SU(2)-connections on E and $f \in C^\infty(M)$ and $\phi \in \Omega^0(E)$.

$$\begin{aligned}(\nabla_1 - \nabla_2)(f\phi) &= \nabla_1(F\phi) - \nabla_2(f\phi)\\ &= df \otimes \phi + f(\nabla_1\phi) - df \otimes \phi - f(\nabla_2\phi)\\ &= f(\nabla_1\phi) - f(\nabla_2\phi) = f(\nabla_1 - \nabla_2)\phi\end{aligned}$$

for any vector field V and $\forall \phi_1, \forall \phi_2 \in \Omega^0(E)$

$$<(\nabla_1 - \nabla_2)_v \phi_1, \phi_2> + <\phi_1, (\nabla_1 - \nabla_2)_v \phi_2> = v<\phi_1,\phi_2> - v<\phi_1,\phi_2> = 0.$$

And clearly $(\nabla_1 - \nabla_2)$ is linear on $\Omega^0(E)$. Thus we have

$$\nabla_1 - \nabla_2 \in \Omega^1(\mathcal{G}_E).$$

REMARK. (i) In terms of local expression the connection $\nabla = d + A$

$$g(A) = g(dg^{-1}) + gAg^{-1}$$

(ii) The induced action on the curvatures $g(R^\nabla) = gR^\nabla g^{-1}$.

(iii) $R^{\nabla+A} = R^\nabla + d^\nabla A + [A,A]$.

On an oriented Riemannian 4-manifold M, the Hodge Star Operator $*: \Lambda^p T^*M \to \Lambda^{4-p}T^*M$ is defined by

(3.8) $$\alpha \wedge *\beta = (\alpha,\beta) d\ \mathrm{vol} \in \Lambda^4 T^*M$$

REMARK. We summerize the $*$-operator on 4-manifold

(i) On Λ^2T^*M, $*^2 = I$, let (x_1,x_2,x_3,x_4) be a local coordinate of a neighborhood of p in M. We have an eigenvalue decomposition

$$\Lambda^2T_p^*M = \Lambda_+^2T_p^*M \oplus \Lambda_-^2T_p^*M$$

where $\Lambda_\pm^2T_p^*M$ = the ±1 - eigenspace of $*$

$$= <dx_1\wedge dx_2 \pm dx_3\wedge dx_4,\ dx_1\wedge dx_3 \pm dx_4\wedge dx_2,$$
$$dx_1\wedge dx_4 \pm dx_2\wedge dx_3>$$

ii) On 2-forms $*$ is conformally invariant.

(iii) On the 4-manifold the adjoint operator $\delta^{\nabla} = -*d^{\nabla}*$.

DEFINITION 3.9. A connection $\nabla \in C$ is self-dual if $*R^{\nabla} = R^{\nabla}$. If $*R^{\nabla} = -R^{\nabla}$ then is called an anti-self-dual connection. Let be the set of all self-dual connections. Let $B = C/\mathcal{G}$ and $M = \mathcal{C}/\mathcal{G}$.

The compatible actions of a finite group G on the bundle $E \xrightarrow{\pi} M$, that is, G-action on E through bundle isomorphism such that π is a G-map, induces the actions of G on C.

DEFINITION 3.10. (i) On $\Omega^0(E)$ $h(\sigma) = h \circ \sigma \circ h^{-1}$ for $\forall h \in G$, $\forall \sigma \in \Omega^0(E)$ where h^{-1} is a diffeomorphism of M and h is a bundle map.

(ii) For $\forall \nabla \in C$ and V is any vector field on M, $\forall \sigma \in \Omega^0(E)$

$$h(\nabla)_V = h[\nabla_V(h^{-1}\sigma)]$$

(iii) On $\Omega^p(\mathcal{G}_E)$, $(h\phi)_{v_1,\cdots,v_p} = h\phi_{h_*^{-1}v_1,\cdots,h_*^{-1}v_p}$, for $h \in G$,

$\phi \in \Omega^p(\mathcal{G}_E)$ and $v_1,\cdots,v_p \in TM$.

REMARK. (i) This group action on the space of connections was first defined and studied by Fintushel and Stern [12].

(ii) In the definition, $(h(\nabla)_V\sigma)_x = (h\nabla_V h^{-1}\sigma)_x = h(\nabla_{V_{h^{-1}(x)}}(h^{-1}\sigma h_{h^{-1}(x)})) \in$

(iii) On the curvature $R^{h(\nabla)} = h \circ R^{\nabla} \circ h^{-1}$.

From the definition (3.10) we have some immediate consequences.

LEMMA 3.11. (i) G <u>acts on</u> C

(ii) G <u>acts on</u> $B = C/\mathcal{G}$

PROOF. (i) is immediate from definitions of connection and definition (3.10).

(ii) For any $h \in G$, $g \in \mathcal{G}$, $v \in T_{(TM)}$ and $\sigma \in \Omega^0(E)$

$$\begin{aligned} h(g(\nabla))_v\sigma &= h[g(\nabla)_v(h^{-1}\sigma h)] \\ &= h[g\nabla_v g^{-1}(h^{-1}\sigma h)] \\ &= (hgh^{-1})[h\nabla_v h^{-1}(hg^{-1}h^{-1})\sigma h] \\ &= (hgh^{-1})[h(\nabla)_v(hg^{-1}h^{-1})\sigma] \\ &= (hgh^{-1})[h(\nabla)]_v\sigma \end{aligned}$$

Thus $h(g(\nabla)) = (hgh^{-1})[h(\nabla)]$.

The definition $h[\nabla] = [h(\nabla)]$ is well-defined on B.

Here G acts on the gauge group $\mathcal{G}$ by conjugation.

REMARK. Since G acts on M by isometries, G-action commutes with the $*$-operator. Thus G preserves the self-dual connections, that is, G acts on $\mathcal{M}$.

CHAPTER IV

THE INDEX OF THE FUNDAMENTAL ELLIPTIC COMPLEX

Let V be a n-dimensional vector space with a inner product $<\,,\,>$ by defining a homomorphism $\Lambda^2(V) \to \mathrm{Hom}(V,V)$ by $(U \wedge V)W = <u,w>v - <v,w>u$ for all $u,v,w \in V$. We have $<(u\wedge v)w_1,w_2> + <w_1,(u\wedge v)w_2> = 0$. We can identify $\Lambda^2(V)$ with the Lie algebra $SO(n)$ of the special orthogonal group $SO(n)$.

On 4-dimension, the decomposition $\Lambda^2 = \Lambda^2_+ + \Lambda^2_-$ corresponds the decomposition of the Lie algebra $SO(4) = SO(3) \oplus SO(3)$. So, we can consider $\Lambda^2_\pm$ 3-dimensional Lie algebras. On the Lie group level the homomorphism π: $\mathrm{Spin}(4) = \mathrm{Spin}(3) \times \mathrm{Spin}(3) \to SO(4)$ defined by $\pi(g,h)x = gxh^{-1}$ has kernel $\{(-1,-1),(1,1)\}$. As a manifold $\mathrm{Spin}(3) = SU(2) = Sp(1) = S^3$, and π is the 2-fold universal covering map. Thus for any oriented Riemannian 4-manifold M, we may have, at least locally, the two complex spinor bundles V_+(even) and V_-(odd). Denote the total spin bundle $V = V_+ \oplus V_-$. The complex endomorphism bundle of V is isomorphic to the complexified Clifford algebra bundle of the cotangent bundle T^*M.

We will need more precise correspondence of this isomorphism $\Lambda_C(T^*M) \cong \mathrm{End}_C(V)$.

$$(4.1)\qquad \begin{cases} \Lambda^1_C(T^*M) \approx \mathrm{Hom}_C(V_+,V_-),\ \Lambda^3_C(T^*M) \approx \mathrm{Hom}_C(V_-,V^+) \\ \Lambda^2_{C_+}(T^*M) \approx \mathrm{Hom}_C(V_+,V_+)^0,\ \Lambda^2_{C_-}(T^*M) \approx \mathrm{Hom}_C(V_-,V_-)^0 \\ \Lambda^0_C(T^*M) \approx \Lambda^4_C(T^*M) \approx C. \end{cases}$$

Where 0 denotes the traceless endomorphisms, and $\Lambda^{\cdot}_C$ denotes the complexification of $\Lambda^{\cdot}$.

Let $E \to M$ be a quaternion line bundle with $k = 1$ over a compact oriented simply connected smooth 4-manifold M. For a self-dual connection $\nabla \in \mathcal{O}\!\mathcal{L}$ there is a most important elliptic complex in our studies which is called the fundamental elliptic complex:

$$(4.2)\qquad 0 \longrightarrow \Omega^0_4(\mathcal{G}_E) \xrightarrow{d^\nabla} \Omega^1_3(\mathcal{G}_E) \xrightarrow{d^\nabla_-} \Omega^2_-(\mathcal{G}_E)_2 \longrightarrow 0$$

Where $\Omega^{\cdot}_k(\mathcal{G}_E)$ is the Sobolev completion of $\Omega^{\cdot}(\mathcal{G}_E)$ with a Sobolev k-norm $\|\phi\|^2_k = \int_M \{\|\phi\|^2 + \cdots + \|\nabla^k\phi\|^2\} d\,\mathrm{vol}$.

REMARK. It is a basic fact that the Sobolev completion of the space of cross sections of a smooth finite dimensional vector bundle is a Hilbert manifold. The operator d^∇ in (4.2) are continuous. The gauge group action $\mathcal{G}$ on the space of connections C extends to a differentiable action of $\mathcal{G}_4$ on C_3. If we do not complete (4.2) with Sobolev norm, then we cannot guarantee the elliptic operators to be invertible. Moreover the index of (4.2) is independent of the k-th Sobolev norm. This fundamental complex was first defined and studied by Atiyah, Hitchin and Singer [1].

LEMMA 4.3. <u>The sequence (4.2) is an elliptic complex with finite dimensional cohomologies</u>.

PROOF. Since the connection ∇ is self-dual $d_-^{\nabla}\circ d^{\nabla} = R_-^{\nabla} = 0$.

For any cotangent vector $\xi \in T_x^*M$ the symbol sequence of (4.2) at is

$$(4.4) \qquad 0 \longrightarrow R \otimes \mathcal{G}_E \xrightarrow{-\Lambda\xi \otimes \mathrm{id}} T_x^*M \otimes \mathcal{G}_E \xrightarrow{P_-(-\Lambda\xi)\otimes \mathrm{id}} \Lambda_-^2 T_x^*M \otimes \mathcal{G}_E \longrightarrow 0$$

where $P_-: \Lambda^2 T_x^*M \to \Lambda_-^2 T^*M$ is the orthogonal projection to the anti-self dual part. Let $\{e_1,e_2,e_3,e_4\}$ be an orthonormal basis for T_x^*M. Then $\{(e_1\wedge e_2 - e_3\wedge e_4),(e_1\wedge e_3 - e_4\wedge e_2),(e_1\wedge e_4 - e_2\wedge e_3)\}$ is an orthonormal basis for $\Lambda_-^2 T_x^*M$. Since $P_-'\{e_1\wedge e_2, e_1\wedge e_3, e_1\wedge e_4\} = \{\frac{1}{2}(e_1\wedge e_2 - e_3\wedge e_4),$ $\frac{1}{2}(e_1\wedge e_3 - e_4\wedge e_2), \frac{1}{2}(e_1\wedge e_4 - e_2\wedge e_3)\}$ and since any non-zero $\xi \neq 0$ in T^*M be a basis member. The sequence

$$0 \longrightarrow R \xrightarrow{\Lambda\xi} T_x^*M \xrightarrow{\Lambda\xi} \Lambda_-^2 T_x^*M \longrightarrow 0$$

is an exact sequence for $\xi \neq 0 \in T^*M$. This sequence is split because it is free. So the tensor product with Lie algebra is also a short exact sequence.

Let G be a finite group. G acts on a quaternion line bundle $E \xrightarrow{\pi} M$ with instanton number one through bundle isomorphism such that π is a G-map. We choose metrics on E and M which are G-invariant. Assume that the connection ∇ is G-invariant self-dual. Replace the

G *be a compact Lie group acting smoothly on* M. *If* $D: \Gamma(E_1) \to \Gamma(E_2)$ *is a* G-*invariant elliptic operator on* M, *then*

$\mathrm{Index}_G D = B_G(\sigma(D))$, *where* B_G *is the topological* G-*index*.

Let $g \in G$ and let $j: M^h \to M$ be the inclusion

LOCALIZATION THEOREM (4.8, [2]) *Let* G *be a topological cyclic group generated by* g *acting smoothly on the compact manifold* M. *Then the homomorphism* $j!: K_G(TM^g) \to K_G(TM)$ *becomes an isomorphism after "localization at* g", *and its inverse* $(j!)^{-1} = \dfrac{j^*}{\Lambda_{-1}(N^g \times C)}$ *where* $\Lambda_{-1}(N^g \times C)$ *is the restriction to* X^g *of the Thom class of the normal bundle* TN^g *in* TX.

THEOREM (4.9, [3]). *Let* G *be a compact Lie group acting on the compact smooth manifold* M, *and let* D *be a* G-*invariant elliptic operator on* M. *Then the* g-*index of* D *is related to the fixed point set* M^g *by the formula*

$$\mathrm{Ind}_g(D) = (-1)^m \frac{Ch_g(j^*\sigma(D))td(T^g \otimes C)}{Ch_g(\Lambda_{-1}N^g \otimes C)}[TM^g]$$

where $m = \dim M^g$, $j: M^g \to M$ *is the inclusion map and* N^g *is the normal bundle of* M^g *in* M.

m will vary from one component to another.

Calculate the G-index of G-invariant Dirac operator D of (4.6). The Abalytic index, $\mathrm{Ind}_G(D) = \mathrm{Ker}\,D - \mathrm{Coker}\,D \in R(G)$ is a virtual representation of G. Theorem 4.7 said that $\mathrm{Ind}_G(D) = B_G(\sigma(D))$ is the topological G-index of the symbol $\sigma(D) \in K_G(TM)$. For the identity element $e \in G$,

$$\mathrm{Ind}_e(D) = B_e(\sigma(D)) = \mathrm{tr}[e\colon B_G(\sigma(D)) \longrightarrow B_G(\sigma(D))]$$

$$= B(\sigma(D))$$

$$= \mathrm{Ch}(V_- \otimes \mathfrak{g}_C)\mathrm{ch}(V_+ - V_-)\mathrm{td}(TM \otimes C)e(TM)^{-1}[M]$$

$$= \mathrm{ch}(V_-)\cdot\mathrm{ch}(g_C)\cdot \hat{A}(M)\cdot[M]$$

where the genus $\hat{A}(M) = \prod_{i=1}^{2} \dfrac{\frac{x_i}{2}}{\sinh\frac{x_i}{2}}$

$$= 1 - \frac{P_1}{24} + \frac{1}{5760}(-4P_2 + 7P_2^2) + \cdots.$$

$$\mathrm{ch}(\mathfrak{g}_C) = \sum_{i=1}^{3} e^{x_i} = 3 + c_1(\mathfrak{g}_C) + \tfrac{1}{2}(c_1^2(\mathfrak{g}_C) - 2c_2(\mathfrak{g}_C))$$

$$\mathrm{ch}(V_-) = 2 + C_1(V_-) + \tfrac{1}{2}P_1(V_-)$$

$$= 2 + \tfrac{1}{2}P_1(V_-).$$

$C_1(V_+) = 0$ because $V\pm$ is SU(2)-bundle.

Thus $\mathrm{Ind}_e(D) = \mathrm{ch}(V_-)\mathrm{ch}(\mathfrak{g}_C)\hat{A}(M)[M]$

$$= [2 - \tfrac{1}{2}P_1(V_-)][3 + \tfrac{1}{2}P_1(\mathfrak{g}_C)][1 - \frac{P_1(TM)}{24}][M]$$

$$= P_1(\mathfrak{g}_C)[M] + 3\ ch(V_-)A^\wedge(M)[M]$$

$$= -8\ C_2(E)[M] + 3(-b_0 + b_1 - b_2^-)$$

$$= 8k - \frac{3}{2}(\chi - \tau)$$

where $k = -C_2(E)[M]$, b_i = the i-th Betti number of M

b_2^- = rank of $H_-^2(M;C)$

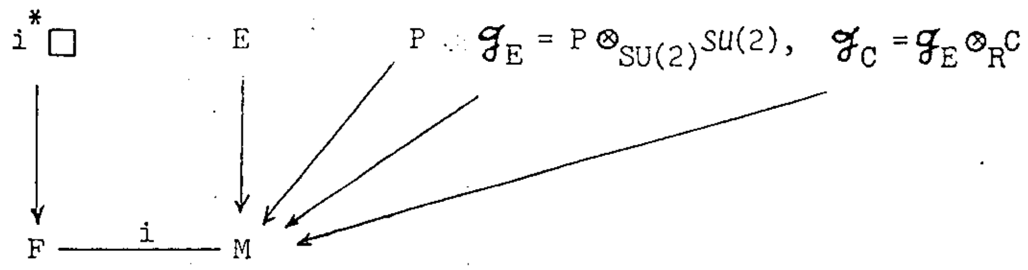
χ = the Euler characteristic of M, τ = the signature of M.

Under our assumption, the instanton number $k = 1$. Since M is simply-connected and intersection form is positive definite, so we have $\chi - \tau = 2$.

LEMMA 4.10. [i]). <u>Let the connection</u> ∇ <u>be</u> G-<u>invariant self-dual and let</u> D <u>be the induced Dirac operator. Then</u> $Ind_e(D) = 5$ <u>where</u> e <u>is the identity element in</u> G.

Let G act smoothly on M^4 and preserve the orientation of M the normal bundle of the fixed point set has even dimensional fibers. The fixed point set M^G is a disjoint union of even dimensional sub-manifolds.

Suppose that a G-action on the bundle $E \to M$ has a fixed point set $F = \{P_i\}_{i=1}^{n_1} \cup \{T^{\lambda_i}\}_{i=1}^{n_2}$ on M where T^{λ_i} is a Riemannian surface with genus λ_i.



where i is the inclusion, P is the associated principal bundle of E. The all induced bundles $i^*\Box \to F$ are SU(2)-bundles. Considering the classifying bundle $SU(2) \to E(SU(2)) \to B(SU(2))$, the classifying space $B(SU(2))$ is 3-connected. The induced bundles on F are trivial because F has at most 2-dimension. The possible actions of h on $i^*E \to F$ are $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, $\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$, $\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$ or $\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$ by considering Z_2-representation on C^2. However $\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$ and $\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$ do not preserve the SU(2)-structure on i^*E. The remainders $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ and $\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$ acts on SU(2) as the usual multiplication of SU(2), and on the associated Lie algebra bundle $\mathfrak{g}_E$ as the adjoint action.

Let $\begin{bmatrix}it & a\\ -\bar{a} & -it\end{bmatrix} \in su(2)$, $\begin{bmatrix}\pm 1 & 0\\ 0 & \pm 1\end{bmatrix}\begin{bmatrix}it & a\\ -\bar{a} & -it\end{bmatrix}\begin{bmatrix}\pm 1 & 0\\ 0 & \pm 1\end{bmatrix} = \begin{bmatrix}it & a\\ -\bar{a} & -it\end{bmatrix}$. So the G-action on $\mathfrak{g}_E \to F$ is trivial and on the complexified Lie algebra bundle over F is also trivial. Thus we have

$$Ch_h(\mathfrak{g}_C) = Ch(\mathfrak{g}_C) = 3 + C_1(\mathfrak{g}_C) + \cdots = 3$$

$$td(TM^h \otimes C) = 1 + \frac{1}{2} C_1(TM^h \otimes C) + \cdots = 1$$

and

$$\frac{Ch_h(V_+ - V_-)Ch_h(V_-)}{Ch_h(\Lambda_{-1}T_P \otimes C)}[P]$$

$$= [\prod^{2} \frac{(e^{\frac{\pi i}{2}} - e^{-\frac{\pi i}{2}})e^{-\frac{\pi i}{2}}}{(1-e^{i})(1-e^{i})}](e^{\pi i} + e^{\pi i})[P] = -\frac{1}{2}$$

Hence $(\mathrm{Ind}_h D) = -\frac{3}{2}$ where $\langle h \rangle = Z_2$. The contribution to the $\mathrm{Ind}_h(D)$ on a fixed point component T^{λ_i} which is a Riemann surface with genus λ_i.

$$\frac{Ch_h(V_+ - V_-)Ch_h(V_-)}{e(T)Ch_h(\Lambda_{-1}N^h \otimes C)}[T^{\lambda_i}]$$

$$= \frac{(e^{\frac{x_1}{2}} - e^{-\frac{x_1}{2}})e^{-\frac{x_1}{2}}(e^{\frac{x_1}{2}+\frac{\pi i}{2}-\frac{x_2}{2}-\frac{\pi i}{2}} - e^{-\frac{x_2}{2}-\frac{\pi i}{2}})}{X_1(1-e^{(x_2+\pi i)})(1-e^{-(x_2+\pi i)})}(e^{x_1} + e^{(x_2+\pi i)})[T^{\lambda_i}]$$

$$= \frac{(1-e^{-x_1})(1+e^{-x_2})(e^{x_1} - e^{x_2})}{X_1(1+e^{x_2})(1+e^{-x_2})}[T^{\lambda_i}]$$

$$= \frac{(1-1+x_1)(1+x_1-1-x_2)}{X_1(1+1+x_2)}[T^{i}]$$

$$= \frac{1}{2}(X_1 - X_2)[T^{\lambda_i}]$$

$$= \frac{1}{2}\; X_1[T^{\lambda_i}] - X_2[T^{\lambda_i}]$$

where X_1 and X_2 represent the Euler classes of the tangent bundle and the normal bundle of T^{λ_i} respectively. Thus

$$\mathrm{Ind}_h(D)|_{T^{\lambda_i}} = -\frac{3}{2}\{X_1[T^{\lambda_i}] - X_2[T^{\lambda_i}]\}$$

THEOREM (4.11 [3]) Let X be a compact oriented manifold of dimension $4k$, and let h be an orientation preserving involution with fixed point set X^h. Let $(X^h)^2$ denote the oriented cobordism class of the self-intersection of X^h in X, then

$$\mathrm{sign}(h{:}X) = \mathrm{sign}[(X^h)^2].$$

In our case the manifold M has dimension 4 with fixed point set $F = \{P_i\}_{i=1}^{1} \cup \{T^{\lambda_i}\}_{i=1}^{n_2}$. The isolate fixed points have the self-intersection 0. For the Riemann surface T^{λ_i} the self-intersection of T^{λ_i} in M is the disjoint union of signed isolated points. Thus

$$\begin{aligned}\mathrm{sign}((T^{\lambda_i})^2) &= \text{the self-intersection of } T^{\lambda_i}\\ &= (X_2)_i[T^{\lambda_i}]\end{aligned}$$

where $(X_2)_i$ is the Euler class of the normal bundle of T^{λ_i} in M.

$$\begin{aligned}\mathrm{sign}(h{:}M) &= \mathrm{sign}((M^h)^2)\\ &= \sum_{i=1}^{n_2} \mathrm{sign}((T^{\lambda_i})^2)\\ &= \sum_{i=1}^{n_2} (X_2)_i[T^{\lambda_i}]\end{aligned}$$

$$\mathrm{Ind}_h(D) = \sum_{i=1}^{n_1} \mathrm{Ind}_h(D)|_{P_i} + \sum_{i=1}^{n_2} \mathrm{Ind}_h(D)|_{T^{\lambda_i}}$$

$$= \sum_{i=1}^{n_1} (-\tfrac{3}{2})[(X_1)[T^{\lambda_i}] - (X_2)_i[T^{\lambda_i}]]$$

$$= -\tfrac{3}{2}\{n_1 + \sum_{i=1}^{n_2} \chi(T^{\lambda_i}) - \sum_{i=1}^{n_2} (X_2)_i[T^{\lambda_i}]\}$$

$$= -\tfrac{3}{2}\{n_1 + \sum_{i=1}^{n_2} \chi(T^{\lambda_i}) - \mathrm{sign}(h:M)\}$$

THEOREM 4.12. Let ∇ be a G-invariant self-dual connection. Let D be the induced Dirac operator by the fundamental elliptic complex (4.5). Let $F = \{P_i\}_{i=1}^{n_1} \cup \{T^{\lambda_i}\}_{i=1}^{n_2}$ be the fixed point set on M. Then we have

$$\begin{cases} \mathrm{Ind}_I(D) = 5 \\ \mathrm{Ind}_h(D) = -\tfrac{3}{2}\{n_1 + \sum_{i=1}^{n_2} \chi(T^{\lambda_i}) - \mathrm{sign}(h:M) \end{cases}$$

Let ∇ be a G-invariant self-dual connection we have a G-invariant elliptic complex $\delta^\nabla + d_-^\nabla \colon \Omega^1(\mathcal{G}_E) \to \Omega^0(\mathcal{G}_E) \oplus \Omega_-^2(\mathcal{G}_E)$. By ellipticity this complex has finite dimensional Ker and Coker. The analytic G-index of this complex = $\mathrm{Ker}(\delta^\nabla + d_-^\nabla) - \mathrm{Coker}(\delta^\nabla + d_-^\nabla)$

$$= H_\nabla^1 - (H_\nabla^0 \oplus H_\nabla^2) \in R(G)$$

where these cohomologies are the cohomologies of (4.2).

The cohomology $H^0_\nabla = 0$ if the connection ∇ is irreducible, otherwise it has dimension one and trivial G-action, since G acts on these cohomology groups.

$$\begin{cases} (\dim H^1_{\nabla_+} + \dim H^1_{\nabla_-}) - (\dim H^0_{\nabla_+} + \dim H^2_{\nabla_+} + \dim H^2_{\nabla_-}) = 5 \\ (\dim H^1_{\nabla_+} - \dim H^1_{\nabla_-}) - (\dim H^0_{\nabla_+} + \dim H^2_{\nabla_+} - \dim H^2_{\nabla_-}) = -\frac{3}{2} A \end{cases}$$

where $A = n_1 + \sum_{i=1}^{n_2} \chi(T^{\lambda_i}) - \operatorname{sign}(h:M)$ and

$\pm$ stands for ± 1 eigenspace of the generator $h \in G$.

THEOREM 4.13. <u>If a connection</u> ∇ <u>is irreducible (reducible) in</u> M, $h(\nabla) = g(\nabla)$, $(hg)^2 = +1$ <u>for some gauge transformation</u> g, <u>then we have</u>

$$\begin{cases} \dim H^1_{\nabla_+} - \dim H^2_{\nabla_+} = \frac{1}{4}(10 - 3A) \\ \dim H^1_{\nabla_-} - \dim H^2_{\nabla_-} = \frac{1}{4}(10 + 3A) \end{cases}$$

<u>where</u> $H^*_{\nabla_\pm}$ <u>is the</u> ± 1 <u>eigenspace of</u> hg.

Each element of the fixed point set M^G in the moduli space M is determined the G-invariant self-dual connections up to the gauge equivalence. We need another index calculation.

Suppose that ∇ is a self-dual irreducible connection such that

$h(\nabla) = g(\nabla)$ for some gauge transformation $g(\neq \pm 1)$ where $\langle h \rangle = G$. Then $(hg)\nabla = \nabla$, $(hg)^2\nabla = \nabla$. We have $(hg)^2 = \pm I \in \mathcal{G}$ If $(hg)^2 = I$, then we have the same result as Theorem 4.12. If $(hg)^2 = -I$, then f has order 4 on the total space E and f has order 2 on the base manifold M, where $f = hg$. Again we have a f-invariant fundamental elliptic complex

$$0 \longrightarrow \Omega^0(\mathcal{G}_E) \longrightarrow \Omega^1(\mathcal{G}_E) \longrightarrow \Omega^2_-(\mathcal{G}_E) \longrightarrow 0$$

As before we have an induced elliptic operator

$$D: \quad (V_+ \otimes V_- \otimes \mathcal{G}_C) \longrightarrow \Gamma(V_- \otimes V_- \otimes \mathcal{G}_C),$$

and its index

$$\operatorname{ind}_f(D) = (-1)^{\frac{\dim M^f}{2}} \frac{Ch_f(V_+ - V_-)Ch_f(V_-)Ch_f(\mathcal{G}_C)td(TM^f \otimes C)}{e(TM^f)\ Ch_f(\Lambda_{-1}N^f \otimes C)}[M^f]$$

$$= (-1)^{\frac{\dim M^h}{2}} \frac{Ch_h(V_+ - V_-)Ch_h(V_-)Ch_f(\mathcal{G}_C)td(TM^h \otimes C)}{e(TM^h)\ Ch_h(\Lambda_{-1}N^h \otimes C)}[M^h]$$

The only difference between this formula and previous formula is that the h-Chern character $Ch_h(\mathcal{G}_C)$ is replaced by f-Chern character $Ch_f(\mathcal{G}_C)$. When we consider the fixed point set $F = M^f = M^h =$ $\{P_i\}_{i=1}^{n_1} \cup \{T^{\lambda_i}\}_{i=1}^{n_2}$ on M. The various associated $SU(2)$ bundles, specially $\mathfrak{g}_C$ on the fixed point set F, are trivial. On E f acts as a multiplication of $\begin{pmatrix} e^{\frac{2\pi i}{4}} & 0 \\ 0 & e^{\frac{-2\pi i}{4}} \end{pmatrix}$ with order 4. On the associated Lie algebra bundle $\mathfrak{g}_E$, f acts adjointly i.e.

$$\begin{pmatrix} e^{i\theta}, & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\begin{pmatrix} it, & a \\ -\bar{a}, & -it \end{pmatrix}\begin{pmatrix} e^{-i\theta}, & 0 \\ 0 & , e^{i\theta} \end{pmatrix} = \begin{pmatrix} it & , & e^{2i\theta}a \\ -e^{-2i\theta}a & & -it \end{pmatrix}.$$

So if we write $\mathfrak{g}_E = \underline{R} \oplus \underline{C}$, then f acts trivially on $\underline{R}$ and f acts with weight 2 on $\underline{C}$. Using splitting principle $Ch_f(\mathfrak{g}_C) = e^{x_1} + e^{x_2}e^{\pi i} + e^{x_3}e^{-\pi i} = 1 - 1 - 1 = -1$. And since $td(TM^h \otimes C) = 1$,

$$\frac{Ch_h(V_+-V_-)Ch_h(V_-)}{Ch_h(\Lambda_{-1}N^h \otimes C)}[P] = -\frac{1}{2} \quad \text{and} \quad \frac{Ch_h(V_+-V_-)Ch_h(V_-)}{e(T^{\lambda_i})Ch_h(\Lambda_{-1}N^h \otimes C)}[T^{\lambda_i}] = \frac{1}{2}\,(\chi(T^{\lambda_i}) - X_2(T^{\lambda_i}))$$

where $X_2(T^{\lambda_i})$ is the self-intersection number of T^{λ_i}. Thus we have

$$\begin{aligned} Ind_f(D) &= \sum_{i=1}^{n_1}[Ind_f(D)]_{P_i} + \sum_{i=1}^{n_2}[Ind_f(D)]_{T^{\lambda_i}} \\ &= \sum_{i=1}^{n_1}\frac{1}{2} + \sum_{i=1}^{n_2}\frac{1}{2}\,(\chi(T^{\lambda_i}) - X_2(T^{\lambda_i})) \\ &= \frac{1}{2}\,\{n_1 + \sum_{i=1}^{n_2}\chi(T^{\lambda_i}) - sign(h:M)\} \end{aligned}$$

Similarly we can calculate $Ind_{f^2}(D)$ and $Ind_{f^3}(D)$.

$Ind_{f^3}(D) = \frac{1}{2}\,\{n_1 + \sum^{n_2} \chi(T^{\lambda_i}$

THEOREM 4.12'. *Let* ∇ *be a self-dual irreducible connection and* $h(\nabla) = g(\nabla)$ *for some gauge transformation* g, $(hg)^2 = -I$, *let* D *be the induced elliptic operator by the fundamental elliptic complex* (4.5). *Let* $F = \{P_i\}_{i=1}^{n_1} \cup \{T^{\lambda_i}\}_{i=1}^{n_2}$ *be the* G-*fixed point set on* M *and let* $f = hg$. *Then we have*

$$\begin{cases} \mathrm{Ind}_{f^0}(D) = 5 \\ \mathrm{Ind}_{f^1}(D) = \frac{1}{2}\{n_1 + \sum_{i=1}^{n_2} \chi(T^{\lambda_i}) - \mathrm{sign}(h:M)\} \\ \mathrm{Ind}_{f^2}(D) = 5 \\ \mathrm{Ind}_{f^3}(D) = \frac{1}{2}\{n_1 + \sum_{i=1}^{n_2} \chi(T^{\lambda_i}) - \mathrm{sign}(h:M)\}. \end{cases}$$

For simplicity let $A \equiv n_1 + \sum_{i=1}^{n_2} \chi(T^{\lambda_i}) - \mathrm{sign}(h:M)$.

Now let us consider the analytic index for the fundamental f-invariant elliptic complex. $\mathrm{Ind}_H(D) = H^1_\nabla - H^2_\nabla \in R(H)$, where $H = \langle f \rangle$. Irreducible $H = \langle f \rangle$-decomposition $H^1_\nabla = \bigoplus_{n=0}^{3} h^1_n H^1_{\nabla\cdot n}$, $H^2_\nabla = \bigoplus_{n=0}^{3} h^2_n H^2_{\nabla\cdot n}$ where h acts as $(i)^n$ on $H^*_{\nabla\cdot n}$ and $h^*_n \in \mathbb{Z}$. Then

$$\begin{cases} \mathrm{Ind}_{f^0}(D) = (h_0^1 + h_1^1 + h_2^1 + h_3^1) - (h_0^2 + h_1^2 + h_2^2 + h_3^2) = 5 \\ \mathrm{Ind}_{f^1}(D) = (h_0^1 + ih_1^1 - h_2^1 - ih_3^1) - (h_0^2 + ih_1^2 - h_2^2 - ih_3^2) = \frac{1}{2} A \\ \mathrm{Ind}_{f^2}(D) = (h_0^1 - h_1^1 + h_2^1 - h_3^1) - (h_0^2 - h_1^2 + h_2^2 - h_3^2) = 5 \\ \mathrm{Ind}_{f^3}(D) = (h_0^1 - ih_1^1 + ih_3^1) - (h_0^2 - ih_1^2 - h_2^2 + ih_3^2) = \frac{1}{2} A \end{cases}$$

From these we obtain

THEOREM 4.13'. Under the hypothesis of Theorem 4.12', then we have

$$\begin{cases} h_0^1 - h_0^2 = \frac{1}{4}(10 + A) \\ h_2^1 - h_2^2 = \frac{1}{4}(10 - A). \end{cases}$$

REMARK. Above calculation $h_1^1 - h_1^2 = 0$ and $h_3^1 - h_3^2 = 0$, actually $h_1^* = h_3^* = 0$ by construction.

Next suppose that ∇ is a self-dual reducible connection such that $h(\nabla) = g(\nabla)$ for some gauge transformation $g \notin \Gamma_\nabla$, where Γ_∇ is the isotropy subgroup of ∇ which is $SO(2)$. Then $(hg)\nabla = \nabla$ and $(hg)^2(\nabla) = \nabla$. So $(hg)^2 \in \Gamma_\nabla$.

Consider the extended gauge group $\mathcal{G}^e = \{g: E \to E \mid g$ is a bundle isomorphism which lies on id or h on $M\}$. Then we have exact sequences

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}^e \longrightarrow Z_2 = \{id, h\} \longrightarrow 0,$$

and $$0 \longrightarrow \Gamma_\nabla \longrightarrow \Gamma_\nabla^e \longrightarrow Z_2 \longrightarrow 0$$

where Γ_∇^e is the isotropy subgroup of ∇ in the extended gauge group $\mathcal{G}^e$. Then Γ_∇^e is either $\Gamma_\nabla \times Z_2$ or $O_2 \simeq \Gamma_\nabla + \sigma\Gamma_\nabla$ where $\sigma = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$. The extended gauge transformation $hg \in \Gamma_\nabla^e$ lies on h.

If $\Gamma_\nabla^e \simeq \Gamma_\nabla \times Z_2$, then $hg = g_1 h$ for some $g_1 \in \Gamma_\nabla$, $hgh = g_1$. Since $(hg)^2\nabla = \nabla$, $(g_1 g)\nabla = \nabla$ and so $g(\nabla) = \nabla$.

This contradicts to $g \notin \Gamma_\nabla$. Thus $\Gamma_\nabla^e \not\simeq \Gamma_\nabla \times Z_2$ of course we may have $\Gamma_\nabla^e \simeq \Gamma_\nabla \times Z_2$, but in this case $g \in \Gamma_\nabla$.

If $\Gamma_\nabla^e \simeq O_2 \simeq \Gamma_\nabla \oplus \sigma\Gamma_\nabla$, then $hg = g_1\sigma g_2$ for some $g_1, g_2 \in \Gamma_\nabla$. $\sigma = g_1^{-1} h g g_2^{-1}$ lies on h. ∇ is σ-invariant. From this expression, that has order 2 it is not clear, but by construction σ has of form $\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$. So σ acts on the Lie algebra bundle $\mathcal{G}_E$ as $\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}it & a\\ -\bar{a} & -it\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}it & -a\\ \bar{a} & -it\end{pmatrix}$. Thus we obtain $Ch_\sigma(D) = -1$.

By similar calculation with the irreducible connection, we have

THEOREM 4.12". *Let ∇ be a self-dual reducible connection and $h(\nabla) = g(\nabla)$ for some gauge transformation $g \notin \Gamma_\nabla$. Let D be the induced elliptic operator by the fundamental elliptic complex (4.5). Let $F = \{P_i\}_{i=1}^{n_1} \cup \{T^{\lambda_i}\}_{i=1}^{n_2}$ be the G-fixed point set on M. Then $hg \in \Gamma_\nabla \oplus \sigma\Gamma_\nabla$ where $\sigma = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$. Moreover if $\sigma = g_1^{-1} h g g_2^{-1}$ for some*

$g_1, f_2 \in \Gamma_\nabla$, then

$$\begin{cases} \operatorname{Ind}_0(D) = 5 \\ \operatorname{Ind}(D) = \frac{1}{2} A. \end{cases}$$

THEOREM 4.13". Under the assumption of Theorem 4.12", we have

$$\begin{cases} \dim H^1_{\nabla_+} - \dim H^2_{\nabla_+} = \frac{1}{4}(14 + A) \\ \dim H^1_{\nabla_-} - \dim H^2_{\nabla_-} = \frac{1}{4}(10 - A). \end{cases}$$

We will see the number $A = n_1 + \sum_{i=1}^{n_2} \chi(T^{\lambda_i}) - \operatorname{sign}(h:M)$ is $+2$ in Theorem 6.6. We can calculate the dimension of the fixed point components of $\mathcal{M}^G$ of the moduli space $\mathcal{M}$, by Theorem 4.13, Theorem 4.13' and Theorem 4.13".

COROLLARY 4.14. Suppose $\nabla \in \mathcal{M}^G$

(i) If ∇ is irreducible, $h(\nabla) = \nabla$, then the dimension of ∇-component is 1

(ii) If ∇ is irreducible $h(\nabla) = g(\nabla)$, $(hg)^2 = -1$, then the dimension of ∇-component is 3

(iii) If ∇ is reducible, then ∇ is a singular cone point of a 1-dimensional fixed point component and a 3-dimensional fixed point component.

CHAPTER V

PERTURBATION OF $\mathcal{M}^G$

Let $\pi\colon E \to M$ be a quaternion line bundle with instanton number one and with a G-action on E through bundle isomorphism such that π is a G-map. Here G is a finite group and M is a simply connected, closed, and smooth 4-manifold with a positive definiti intersection form. The moduli space $\mathcal{M}$ which is the set of self-dual connections $\mathcal{O}\!\mathcal{L}$ on E modulo the group $\mathcal{G}$ of gauge trnasformations, is a G-space but may not be a manifold, when we start with G-invariant metric on M. In this chapter we would like to find a G-invariant metric on M such that the fixed point set $\mathcal{M}^G$ in the moduli space $\mathcal{M}$ is a smooth set which consists of smooth manifolds. To do this we will use the Uhlenbeck argument which was used to find generic metrics such that the moduli space of irreducible self-dual connections is a manifold.

Let $C^k = C^k(GL(TM))$ be the set of C^k-automorphisms of the tangent bundle, that is, the group of gauge transformations for the bundle of frames. Then C^k is a Banach manifold. If g is a fixed G-invariant metric on M, and $\phi \in C^k$ then the pullback-action on $(T^*M \otimes T^*M)$ by ϕ gives a new metric $\phi^*(g)$. Every metric is realized in this way. On each fiber considering $O(n) \to GL(n) \to \text{sym}(n) = \frac{GL(n)}{O(n)}$. The metrics on M is uniquely determined in this fashion if $\phi^*_x \in \text{sym}(n)$ for each

$x \in M$. Thus many different ϕ's $\in C^k(GL(TM))$ may produce the same metric on M.

Let $P_-: \Omega^2 \to \Omega^2_-$ be the projection onto the anti-felf-dual 2 forms with respect to the metric g. Then $\phi^* P_- \phi^{-1*}$ is the projection onto anti-self-dual 2 forms with respect to themetric $\phi^*(g)$, that is, the following diagram commutes

$$\begin{array}{ccc}
\Gamma(\Lambda^2 T^* M)_g & \xrightarrow{P_-} & \Gamma(\Lambda^2_- T^* M)_g \\
\uparrow \phi^{-1*} & & \uparrow \phi^* \\
\Gamma(\Lambda^2 T^* M)_{\phi^*(g)} & \xrightarrow{P_-} & \Gamma(\Lambda^2_- T^* M)_{\phi^*(g)}
\end{array}$$

where P_- is the projection onto the anti-self-dual 2-forms with respect to the metric $\phi^*(g)$.

Let k be large enough and define

$$\Phi: \hat{C}_{\ell-1} \times C^k \longrightarrow \Omega^2_-(\ \ E)_{\ell,2}$$

by
$$\Phi(\nabla,\phi) = P_-(\phi^{-1*} R^\nabla),$$

where $\hat{C}_{\ell-1}$ is the set of irreducible connections on E with $(\ell-1)$-Sobolev norm. $\Phi(\nabla,\phi) = 0$ if and only if R^∇ is self-dual with respect to $\phi^*(g)$. Thus C^k is chosen as our parameter space precisely so that we could detect self-duality by mapping into a fixed space $\Omega^2_-(\mathcal{G}_E)_{\ell-2}$ with respect to the metric g.

LEMMA 5.1. *This map* $\Phi: \hat{C}_{\ell-1} \times C^k \to \Omega^2_-(\mathcal{G}_E)_{\ell-2}$ *is a G-map.*

PROOF. For any $h \in G$, and any $(\nabla,\phi) \in C^{\hat{}} \times C^k$

$$\begin{aligned}(h(\nabla),h(\phi)) &= P_-[(h(\phi)^{-1*}R^{h(\nabla)}] \\ &= P_-[h(\phi)^{-1*}, hR^{\nabla}h^{-1}] \\ &= P_-h[\phi^{-1*}R^{\nabla}] \\ &= h\ P_-[\phi^{-1*}R^{\nabla}] \\ &= h\ \Phi(\nabla,\phi)\end{aligned}$$

Fourth equality holds because the metric g is G-invariant. Thus we have a G-invariant map Φ.

COROLLARY 5.2. *A connection* ∇ *is self-dual with respect to* $\phi^*(g)$ *if and only if* $h(\nabla)$ *is self-dual with respect to* $(h\phi)^*(g)$.

THEOREM (5.3. [14]). *The map* Φ *is smooth and has zero as a regular value.*

Since zero is a regular value $\Phi^{-1}(0)$ is an infinite dimensional Banach manifold of self-dual connections parametrized by the set of all metric C^k. Since the gauge transformation group $\mathcal{G}_\ell$ acts on M trivially, and $\mathcal{G}_\ell$ acts on $\Phi^{-1}(0)$.

THEOREM (5.4. [14]). $\phi^{-1}(0)/\mathcal{G}_\ell \subset (\hat{C}_{\ell-1}/\mathcal{G}_\ell \times C^k)$ <u>is a manifold</u>.

We have the following diagram

$$
\begin{array}{ccccc}
 & & \phi^{-1}(0) & \longrightarrow \hat{C}_{\ell-1} \times C^k & \longrightarrow \Omega^2_-(\mathcal{G}_E)_{\ell-2} \\
 & & \downarrow & \downarrow & \\
[\phi^{-1}(0)/\mathcal{G}_\ell]^G & \longrightarrow & \phi^{-1}(0)/\mathcal{G}_\ell & \longrightarrow \hat{C}_{\ell-1}/\mathcal{G}_\ell \times C^k & \\
\downarrow \bar{\bar{\pi}} & & \downarrow \bar{\pi} & \downarrow \pi & \\
[C^k]^G & \longrightarrow & C^k & \longrightarrow C^k &
\end{array}
$$

Here $\phi^{-1}(0)/\mathcal{G}_\ell$ is the parametrized moduli space by the metrics C^k. For each metric $\phi \in C^k$, $\pi^{-1}(\phi) = \hat{M}^*_{\phi^*(g)}$ is the moduli space of irreducible connections with respect to the metric $\phi^*(g)$. As a set $\phi^{-1}(0)/\mathcal{G}_\ell = \bigcup_{\phi\in C} \hat{M}^*_{\phi^*(g)}$.

THEOREM 5.5. <u>The manifold</u> $\phi^{-1}(0)/\mathcal{G}_\ell$ <u>is a</u> G-<u>space.</u>

PROOF. Since ϕ is a G-map, $\phi^{-1}(0)$ is a G-space. By Corollary 5.2, a connection ∇ is self-dual with respect to $\phi^*(g)$ if and only if $h(\nabla)$ is self-dual with respect to metric $(h\cdot\phi)^*(g)$.

For any gauge transformation $g \in \mathcal{G}$, $h \in G$, ∇ $\hat{M}^*_{\phi^*(g)}$

$$
\begin{aligned}
h[g(\nabla)] = h[g\nabla g^{-1}] = hg\nabla g^{-1}h^{-1} &= (hgh^{-1})(h\nabla h^{-1})(hg^{-1}h^{-1}) \\
&= h(g)\cdot[h(\nabla)]
\end{aligned}
$$

Since G acts on $\mathcal{G}$ by conjugation, $h(g) \in \mathcal{G}$. Hence the map $\hat{M}_{\psi^*(g)} \longrightarrow \hat{M}_{(h\phi)^*(g)}$ given by $[\nabla] \to [h(\nabla)]$ is well-defined.

Thus the G-action on $\Phi^{-1}(0)/\mathcal{G}_\ell$ is well defined.

Since the projection map $\pi: C_{\ell-1}/\mathcal{G}_\ell \times C^k \to C^k$ is a G-map and the restriction $\bar{\pi}: \Phi^{-1}(0)/\mathcal{G}_\ell \to C^k$ is also a G-map. In [14] they showed the map $\bar{\pi}^{-1}(\phi) = \hat{M}_{\phi^*(g)}$ which has dimension 5.

The map $\bar{\pi}: \Phi^{-1}(0)/\mathcal{G}_\ell \;\; C^k$ is a G-Fredholm map. The restriction map $\bar{\bar{\pi}}: (\Phi^{-1}(0)/\mathcal{G}_\ell)^G \to (C^k)^G$ is a G-trivial Fredholm map by Theorem 4.13. By Sard-Smale for a Fredholm map between paracompact Banach manifolds we have the following.

THEOREM 5.6. There exists a Baire set of $(C^k)^G$ such that $(\bar{\bar{\pi}})^{-1}(\phi) = (\hat{M}_{\phi^*(g)})^G$ is a smooth manifold in the moduli space $\hat{M}_{\phi^*(g)}$ of the irreducible self-dual connections for the metric $\phi^*(g)$ on M.

From now on we will fix a G-invariant metric on M and we will fix G-invariant metric on the total space E of the bundle such that the fixed point set $\hat{M}^G$, in the moduli space $\hat{M}$ of the irreducible connections, is a manifold. Note that the above Baire set of $(C^k)^G$ is a open dense set for each k.

CHAPTER VI

PERTURBATION IN A NEIGHBORHOOD OF $\mathcal{M}^G$

In Chapter V, we showed that there is a G-invariant generic metric on M such that the fixed point set $\hat{\mathcal{M}}^G$ in the moduli space $\hat{\mathcal{M}}$ of irreducible connections where G is any finite group. We will fix this G-invariant metric and $G = Z_2$. In this chapter we will find the condition which we can locally perturbate in a neighborhood of $\mathcal{M}^G$ by using the result of Chapter IV. Also we will find the condition which we can globally perturbate in a neighborhood of $\mathcal{M}^G$ by using the result of Chapter IV and Petrie's G-transversality argument.

Recall the local structure of the moduli space $\mathcal{M} = \mathcal{C}/\mathcal{G} \subset B$ suppose that the fundamental elliptic complex:

$$0 \longrightarrow \Omega^0_4(\mathfrak{g}_E) \underset{\delta^\nabla}{\overset{d^\nabla}{\rightleftarrows}} \Omega^1_3(\mathfrak{g}_E) \xrightarrow{d^\nabla_-} \Omega^2_{-2}(\mathfrak{g}_E) \longrightarrow 0$$

has the indicated Sobolev norms where $\nabla \in \mathcal{M}$. A connection ∇ is reducible iff $\dim_{\mathbb{R}}(\mathrm{Ker}\, d^\nabla) = 1$ at $\Omega^0(\mathfrak{g}_E)$ iff an isotropy group $\Gamma_\nabla = \{g \in \mathcal{G} \mid g^\nabla g^{-1} = \nabla\} = u(1)$.

Considering an orthogonal decomposition

$$T_\nabla C = \Omega^1_3(\mathfrak{g}_E) = (\mathrm{Im}\, d^\nabla) \oplus (\mathrm{Ker}\, \delta^\nabla)$$

For each $\nabla \in B$ we have a neighborhood of the form

$$(6.1) \qquad \begin{cases} \mathcal{O}_{\nabla,\varepsilon} = \{\nabla + A \mid \delta^{\nabla} A = 0,\ \|A\|_3 < \varepsilon\}, & \text{if } \nabla \text{ is irreducible} \\ \mathcal{O}_{\nabla,\varepsilon}/u(1), & \text{if } \nabla \text{ is reducible.} \end{cases}$$

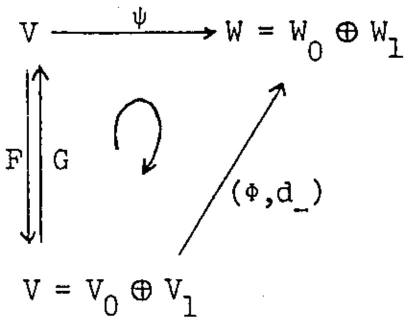
In particular the space $\hat{B}$ of irreducible connections is open in B and is a smooth Hilbert manifold. In a reducible self-dual case the bundle $E = \ell \oplus \bar{\ell}$ splits where ℓ is a complex line bundle on M and the reducible connection $\nabla = \nabla_1 \oplus \bar{\nabla}_1$. The bundle $\Omega^n(\mathcal{G}_E) = \Omega^n \oplus \Omega^n(\ell^2)$ by scalar multiplication. Thus H^1_∇ and H^2_∇ are finite dimensional complex vector spaces.

For a gauge transformation $g \in \mathcal{G}$, the anti-self-dual part $R_-^{g(\nabla)} = g \circ R_-^{\nabla} \circ g^{-1}$. This gives a cross section of the fibration $F \equiv C \times_{\mathcal{G}} \Omega^2_-(\mathcal{G}_E) \to B = C/\tilde{\mathcal{G}}$ where $\tilde{\mathcal{G}}$ acts on $\Omega^2_-(\mathcal{G}_E)$ by adjoint. Namely, the cross section

$$\Psi\colon B = C/_{\mathcal{G}} \longrightarrow C \times_{\mathcal{G}} {}^2_-(\mathcal{G}_E) \text{ is given by } \Psi(\nabla) = (\nabla, R_-^{\nabla}).$$

Let $\nabla \in M$ be a self-dual connection on E. Set $V = \operatorname{Ker}\delta^{\nabla} \subset \Omega^1_3(\mathcal{G}_E)$ and $W = \Omega^2_{-2}(\mathcal{G}_E)$. Define a smooth map $\psi\colon v \to w$ by $\psi(A) = d_-^{\nabla} A + [A,A]_-$. Then the differential $(d\psi)_0 = d_-^{\nabla}\colon V \to W$. Since ψ is a Fredholm map. By setting $V_0 = \operatorname{Ker} d_-^{\nabla}$, $W_0 = \operatorname{coker} d_-^{\nabla}$, $d_-^{\nabla}\colon V = V_0 \oplus V_1 \to W = W_0 \oplus W_1$ and the restriction map $d_-^{\nabla}\colon V_1 \to W_1$ is a Hilbert space isomorphism. Define a map $F\colon V \to V$ by $F = \mathrm{id} + (d_-^{\nabla})^{-1} \circ P_1 \circ (\psi - (d^{\nabla})_0)$ where $P_1\colon W \to W_1$ is the projection. Then $(dF)_0 = \mathrm{id}, F$ has a local inverse

G around 0. Let U be a small neighborhood of 0 on which G is defined. Define $\Phi\colon U \to W_0$ by $\Phi = P_0(\psi - d\psi_0)G$. Then $\Phi(0) = 0$ Φ is commutative with $U(1)$-action we have a local commutative diagram.



We have local coordinates of the moduli space M.

$$(6.2)\qquad \begin{cases} M \cap \mathcal{O}_{\nabla,\varepsilon} \approx \Phi^{-1}(0) & \text{if } \nabla \text{ is irreducible} \\ M \cap (\mathcal{O}_{\nabla,\varepsilon}/U(1)) \approx \Phi^{-1}(0)/U(1) & \text{if } \nabla \text{ is reducible} \end{cases}$$

Let a connection ∇ be a self-dual G-invariant connection considering the fundamental elliptic complex

$$0 \longrightarrow \Omega^0(\mathfrak{g}_E) \underset{\delta^\nabla}{\overset{d^\nabla}{\rightleftarrows}} \Omega^1(\mathfrak{g}_E) \xrightarrow{d_-^\nabla} \Omega^2_-(\mathfrak{g}_E) \longrightarrow 0$$

We have some immediate consequences

LEMMA 6.3.

(i) The generalized connection $d^\nabla: \Omega^p(\mathcal{G}_E) \to \Omega^{p+1}(\mathcal{G}_E)$ is also G-invariant

(ii) The adjoint operator δ^∇ is G-invariant

(iii) The map $\psi: V \to W$ given by $\psi(A) = d_-^\nabla A + [A,A]_-$ is G-invariant

(iv) The map $F: V \to V$ given by $F = id + (d_-^\nabla)^{-1} p_1(\psi - d\psi_0)$ is G-invariant

(v) $\phi = p_0[\psi - d\psi_0]G$ is G-invarinat where the function G is a local inverse of F.

PROOF.

(i) For any $h \in G, \phi \in \Omega^p(\mathcal{G}_E)$ and $v_0 \cdots\cdot v_p \in TM$

$$(d^\nabla h\phi)_{v_0\cdots\cdot v_p} = \sum_{j=0}^{p} (-1)^j \nabla_{v_j}[(h\phi)(v_0\cdots\hat{v_j}\cdots\cdot v_p)]$$

$$+ \sum_{i<j} (-1)^{i+j} (h\phi)([v_i,v_j],v_0\cdots\hat{v_i}\cdots\hat{v_j}\cdots v_p)$$

$$= \sum_{j=0}^{p} (-1)^j \nabla_{v_j} h(\phi_{h_*^{-1}v_0,\cdots h_*^{-1}\hat{v}_j,\cdots h_*^{-1}v_p})$$

$$+ \sum_{i<j} (-1)^{i+j} h(\phi_{[h_*^{-1}v_i,h_*^{-1}v_j]h_*^{-1}v_0\cdots h_*^{-1}\hat{v_i}\cdots h_*^{-1}\hat{v_j}\cdots h_*^{-1}v_p})$$

$$= h(\sum_{j=0}^{p} (-1)^j \nabla_{h_*^{-1}v_j} \phi_{h_*^{-1}v_0\cdots h_*^{-1}\hat{v}_j\cdots h_*^{-1}v_p} +$$

Suppose that a connection ∇ is G-invariant, reducible self-dual. In the fundamental elliptic complex the cohomology groups are $H^0 = R^1$, $H^1_\nabla \simeq C^{k+3}$ and $H^2_\nabla \simeq C^k$. They have G-actions. Also the isotropy group $\Gamma_\nabla \simeq U(1)$ of ∇ in the gauge transformation group $\mathcal{G}$ acts on the cohomology groups H^1_∇ and H^2_∇ by scalar multiplicat Of course H^0_∇ is a trivial representation of G. On the cohomologies H^1_∇ and H^2_∇ the G-action and Γ_∇-action are commutative because their representations are just complex number multiplications.

LEMMA 6.4. On H^1_∇ and H^2_∇ the G-action and $\Gamma_\nabla = U(1)$-action commute.

THEOREM 6.5. [9]: There is an open set M_{λ_0} of the moduli space M of self-dual connections which is a smooth 5-manifold diffoemorphic to $M \times (0,\lambda_0)$ for small $\lambda_0 > 0$ and the complement $k = M \backslash M_{\lambda_0}$ is compact and $\psi(\nabla) \equiv (\nabla, R_-^\nabla)$ is transversal to M_{λ_0}.

From Theorem 6.5, the end part of the moduli space M is naturally diffeomorphic to $M \times (0,\lambda_0)$ for small $\lambda_0 > 0$, and which is formed by irreducible self-dual connections. In our assumption the fixed point set $F = \{p_i\}_{i=1}^{n_1} \cup \{T^{\lambda_i}\}_{i=1}^{n_2}$ on M, where T^{λ_i} is a Riemann surface with genus λ_i. At the end part of the moduli space M has

$$F \times (0,\lambda_0) = \{p_i \times (0, \lambda_0)\}_{i=1}^{n_1} \cup \{T^{\lambda_i} \times (0,\lambda_0)\}_{i=1}^{n_2}$$

as the fixed point components. While the Theorem 4.13 say that some fixed point component in M has dimension $\frac{1}{4}(10 - 3A)$, the Theorem 4.1 say that another fixed point component in M has dimension $\frac{1}{4}(10 + A)$ where $A = n_1 + \sum_{i=1}^{n_2} (T^{\lambda_i}) - \text{sign}(h:M)$. If we compare these results, we have

THEOREM 6.6. <u>Under our basic assumption on</u> $Z_2 = \langle h \rangle$ <u>bundle</u> $E \to M$, <u>we have</u> $A = n_1 + \sum_{i=1}^{n_2} \chi(T^{\lambda_i}) - \text{sign}(h:M) \equiv +2$.

REMARK. This result may be got from the leftschetz fixed point theorem $L(h) = \chi(M^{Z_2})$.

THEOREM 6.7. <u>Suppose that</u> ∇ <u>is</u> G-<u>invariant, reducible self-dual in</u> M, <u>then there is a</u> G-<u>equivariant perturbation around</u> ∇ <u>in</u> B <u>such that the perturbed moduli space</u> M_1 <u>has a neighborhood at</u> ∇ <u>which is an open cone on</u> Cp^2, <u>here</u> ∇ <u>is the cone point which is fixed by</u> G.

PROOF. By Lemma 6.3 the differential map
$\psi: V \equiv \text{Ker}(\delta^\nabla) \subset \Omega^1(\mathcal{G}_E) \quad W = \Omega^2_-(\mathcal{G}_E)$ given by $\psi(A) = d^\nabla_- A + [A,A]_-$ decomposes as a map $(\phi, d^\nabla_-): H^1 \oplus V_1 \to H^2 \oplus W_1$ by a diffeomorphic G-invariant change. The restriction map $d^\nabla_-|_{V_1}$ is a Hilbert space isomorphism and ϕ, d^∇_- are G-invariant.

By Theorem 4.13 and Theorem 6.6

$$\begin{cases} \dim H^1_{\nabla_+} - \dim H^2_{\nabla_+} = 2 \\ \dim H^1_{\nabla_-} - \dim H^2_{\nabla_1} = 4 \end{cases}$$

If $h(\nabla) = g(\nabla)$ for some gauge transformation $g \notin \Gamma_\nabla$, then by Theorem 4.13"

$$\begin{cases} \dim H^1_{\nabla_+} - \dim H^2_{\nabla_+} = 4 \\ \dim H^1_{\nabla_-} - \dim H^2_{\nabla_-} = 2 \end{cases}$$

The map Ψ is a G-equivariant submersion if an only if the map Φ is a G-equivariant submersion. We can easily perturbate Φ into a G-equavariant submersion. For example the first case, by Schur's Lemma map Φ is decomposed as follows

$$\Phi\colon H^1_\nabla = H^1_{\nabla_+} \oplus H^1_{\nabla_-} = C^{k_1+1} \oplus C^{k_2+2} \longrightarrow H^2_\nabla = H^2_{\nabla_+} \oplus H^2_{\nabla_-} = C^{k_1} \oplus C^{k_2}$$

From this decomposition we can choose a map $h\colon H^1_\nabla \to H^2_\nabla$ which is linear surjective and G-invariant. We choose a smooth cutoff function $\rho \in C_0(\mathcal{O}_{\nabla.\varepsilon})$ such that $\rho \equiv 1$ near 0. Then $\Phi + \rho(h - \Phi)\colon H^1_\nabla \to H^2_\nabla$ has a C-linear surjective derivative h at the zero. By (6.2) the new zero set modulo Γ_∇ is a cone on Cp^2.

Suppose that a reducible self-dual connection ∇ is not G-invariant. we can choose an open neighborhood $\mathcal{O}_{\nabla\cdot\varepsilon}/U(1)$ in B such that $(\mathcal{O}_{\nabla\cdot\varepsilon}/U(1)) \cap h(\mathcal{O}_{\nabla\cdot\varepsilon}/U(1)) = \emptyset$. We will see the connection $h(\nabla)$ is also self-dual reducible. Since $h(\delta^{\nabla}A) = \delta^{h(\nabla)}(H(A))$ we have a map $h\colon \mathrm{Ker}\,\delta^{\nabla} \to \mathrm{Ker}(\delta^{h(\nabla)})$.

$$h(\psi(A)) = h[d_-^{\nabla}A + [A,A]_-] = d_-^{h(\nabla)}(hA) + [hA,hA]_-.$$

Thus we have a commutative diagram

$$\begin{array}{ccc}
\ker \delta^{\nabla} & \xrightarrow{\ \psi\ } & \Omega_-^2(\mathcal{G}_E) \\
\Big\downarrow h & & \Big\downarrow h \\
\mathrm{Ker}\, \delta^{h(\nabla)} & \xrightarrow{\ \psi\ } & \Omega_-^2(\mathcal{G}_E)
\end{array}$$

By our usual technique $\psi \simeq (\phi, d_-^{\nabla})\colon H_{\nabla}^1 \oplus V_1 \to H_{\nabla}^2 \oplus W_1$. The restriction $\phi\colon H_{\nabla}^1 \to H_{\nabla}^2$ is a Γ_{∇}-map. The action on the isotropy group $h\colon \Gamma_{\nabla} \to \Gamma_{h(\nabla)}$ is a diffeomorphism. After compact perturbations we have

THEOREM 6.8. *h: [cone on Cp^2 at $[\nabla]$] $\to$ [cone on Cp^2 at $h(\nabla)$] is a diffeomorphism except the cone point* $[\nabla]$.

We set $\lambda = \frac{1}{2}\,\#\{u \in H^2(M:Z) \mid u\,u = 1\}$. We have λ-reducible self-dual gauge equivalence classes $[\nabla^1],\cdots,[\nabla^{\lambda}]$. So far smooth G-manifold M^G

and {cones on Cp^2} is established with λ-singularities. We have a compact perturbation $\psi_1 = \psi + \sigma$ such that $\psi_1 = \psi$ outsides small cone-neighborhoods of the ∇^i's. The differential $d\psi_1$ is also a Fredholm operator which has the same index with $d\psi$.

Next we would like to perturb the new moduli space

$$M_1 = \{\nabla \in B: \psi_1(\nabla) = 0\}, \quad \text{where} \quad \psi_1 = \psi + \sigma: B \longrightarrow C \times_{\mathcal{G}} \Omega^2_-(\mathcal{G}_E)$$

G-equivariantly to a smooth 5-manifold with λ-singularities. Each singularities has a cone neighborhood on Cp^2.

We have the smooth part $M_{\lambda_0} \cup \hat{M}^G \cup$ {open cone on Cp^2 at each reducible connections} in the moduli space $M_1 \subset B$ with λ-singularities We would like to perturb a small neighborhood of $\hat{M}^G$ first locally and then globally by using Petrie G-transversality argument.

Suppose a connection ∇ is G-invariant, (self-dual) irreducible and $\psi_1(\nabla) = 0$. Locally the map ψ_1: $V = \text{Ker}\ \delta^\nabla \to W = \Omega^2_-(\mathcal{G}_E)$ is a Fredholm operator $(d\psi_1)_0 : V \to W$ has the index 5 with splitting $V = \text{Ker}(d\psi_1)_0 \oplus V_1$, $W = \text{coker}(d\psi_1)_0 \oplus W_1$ and $\text{Ker}(d\psi)_0 = R^{k+5}$, $\text{coker}(d\psi_1)_0 = R^k$. The restriction map $(d\psi_1)_0|_{V_1}$ is a Hilbert isomorphism.

REMARK. Here we may not consider the cohomology groups of the fundamental elliptic complex because ∇ may not be a self-dual connection. However ψ_1 is a compact perturbation of ψ. They have the same index and outside of the perturbation they are the same operators in this case we can consider the fundamental elliptic complex.

To see the local structure at irreducible connection ∇ we would like to use the Kuraniski argument for this Fredholm map $\psi_1: V \to W$. Define a differentiable map $F = id + (d\psi_1)^{-1} \circ p_1 \circ (\psi - d\psi_1): V \to V$ where $p_1: W \to W$ be the orthogonal projection. Then $dF = id$. So F is diffeomorphic around the zero. Define a map $Q: Ker(d\psi_1)_0 \to coker(d\psi_1)_0$ by $Q = p_0 \circ \psi \circ F^{-1}$ around the zero, where $p_0: W \to coker(d\psi_1)_0$ is the orthogonal projection. By Lemma 6.3 these all maps are G-equivariant. So the map

$$(Q, (d\psi_1)_0): Ker(d\psi_1)_0 \oplus V_1 \longrightarrow coker(d\psi_1)_0 \oplus W_1$$

is smooth G-equivariant and $\psi_1 = (Q(d\psi_1)_0) \circ F$ is G-equivariant decomposition. We would like to perturb the map

$$Q: Ker(d\psi_1)_0 \longrightarrow coker(d\psi_1)_0$$

to be a map whose derivative is surjective and G-equivariant.

For $\nabla \in M^{\hat{}G}$, $h(\nabla) = g(\nabla)$, if $(hg)^2 = 1$, then since $A = 2$ by Theorem 4.13

$$\begin{cases} \dim H^1_{\nabla_+} - \dim H^2_{\nabla_+} = 1 \\ \dim H^1_{\nabla_-} - \dim H^2_{\nabla_-} = 4. \end{cases}$$

if $(hg)^2 = -1$, then by (4.13') we have

$$\begin{cases} \dim H^1_+ - \dim H^2_+ = 3 \\ \dim H^1_- - \dim H^2_- = 2. \end{cases}$$

The map ψ is a G-equivariant submersion if and only if the map Q is a G-equivariant submersion. In general Q is not a submersion. By the Shur's Lemma, the G-equivariant map Q splits as following.

$$(6.9)\begin{cases} \text{(i)} \quad \text{If } (hg)^2 = 1, \text{ then } Q\colon H^1_\nabla = R^{k+5} = (R^{k_1+1})_+ \oplus (R^{k_2+4})_- \to \\ \qquad H^2_\nabla = R^k = R_+^{k_1} \oplus R_-^{k_2} \\ \text{(ii)} \quad \text{If } (hg)^2 = -1, \text{ then } Q\colon H^1_\nabla = R^{k+5} = (R^{k_1+3})_+ \oplus (R^{k_2+2})_- \to \\ \qquad H^2_\nabla = R^k = R_+^{k_1} \oplus R_-^{k} \end{cases}$$

From this splitting we can easily choose a map $h\colon R^{k+5} \to R^k$ which is a G-invariant epimorphism. Choose a smooth cutoff function $\rho \in C_0(\mathcal{O}_{\nabla\cdot\varepsilon})$ with $\rho \equiv 1$ near 0. Then the map $(1 - \rho)Q + \rho h\colon R^{k+5} \to R^k$ is G-equivariant and its derivative is a epimorphism near 0.

THEOREM 6.10. <u>If a connection ∇ is G-invariant, self-dual and irreducible in $\mathcal{M}$, then there is a G-invariant smooth compact perturbation around ∇ such that the perturbed new moduli space has a smooth 5-dimensional neighborhood at</u> ∇.

PROOF. By above construction and replacing ψ by $[(1 - \rho)Q + \rho h, (d\psi_1)_0]$, we have the result.

We showed that we can perturb locally at each G-invariant self-dual connection into a G-invariant manifold. We now would like to find the condition under which we can perturb a neighborhood of the fixed point set M^G into a G-equivariant smooth neighborhood of M^G. To do this we will introduce Petrie's G-transversality argument and then will apply to our case.

The G-transversality argument gives a solution in terms of an obstruction theory and by giving a criterion for the vanishing of the obstructions.

We would like to introduce two basic ideas. The first is that the problem of equivariant transversality is involved with global phenomena in contrast to the non-equivariant situation is local. The second is that Shur's Lemma applied to the equivariant vector bundles involved with transversality gives a splitting of the problems into two parts. We can solve the first part by using the Thom transversality theorem for the case by trivial group action and the G-homotopy extension theorem. This fixed point part was already done by using generic metric on M. So our main interesting part is the second part, namely the transversality obstruction.

More precisely, the three smooth G-manifolds N, M and Y are given with $Y \subset M$ an G-invariant submanifold and a proper G-map $f: N \to M$ is transverse to Y with $X = f^{-1}(Y)$ and $H \subseteq G$. Then f^H is transversal to $Y^H \subset M^H$ and the normal bundle $\nu(X,N)$ of X in N has a splitting $\nu(X,N)^H \oplus \nu(X,N)_H$ and $\nu(X,N)^H = \nu(X^H,N^H)$ and $\nu(X,N)_H|_{X^H} \oplus \nu(X^H,N)$ the fact that f is transverse to Y throughout

X^H is thus expressed by two equations

$$\begin{cases} (1) \quad \nu(X^H, N^H) = (f^H)^* \nu(Y^H, M^H) \\ (2) \quad \nu(N^H, N)|_{X^H} = (f^H)^* \nu(Y, M)_H \end{cases}$$

By Shur's lemma, the first equation (1) only depend on $f^H: N^H \to M^H$ and is concerned with the action of the normalizer of H mod H on N^H and M^H which by induction can be assumed to act freely. As we know the problem of f^H being transverse to Y^H in M^H by treated by Thom transversality and in particular gives $X^H = (f^H)^{-1}(Y^H)$ as a submanifold of N. Then it is equation (2) which provides the basis for the transversality obstruction theory. Define the G-fiber bundle $V_{\xi,\eta} = \mathrm{Hom}^S(\xi,\eta)$ of real surjective homomorphisms of the G-vector bundle ξ over Y onto the G-vector bundle η over Y. The action of G is defined by conjugation on $V_{\xi,\eta}$. Then $V^H_{\xi,\eta}$ is a G/H fiber bundle over Y^H if H is normal in G. The fiber over $y \in Y^H$ is $V(H)_y \equiv \mathrm{Hom}^S_H(\xi_y, \eta_y)$ the space of real surjective H-homomorphisms from the fiber ξ_y to η_y. Then Petrie show that,

THEOREM (6.11. [22] G-transversality Theorem). <u>Let</u> $f: N \to M$ <u>be transverse to</u> Y <u>on</u> $Z_{h-1} = \bigcup_{k>h} N^k$ <u>and without loss of generality suppose</u> $f^H \not\pitchfork Y^H$. <u>Let</u> $X^k = (f^k)^{-1}(Y^k)$, $K \geq H$, <u>and</u> $X_H = \bigcup_{K>H} X^k$. <u>Then there is a</u> G-<u>invariant neighborhood</u> W <u>of</u> Z_{h-1} <u>and a proper</u> G-<u>homotopy</u>

of f rel $W \cup Z_H$ to a map $Q \pitchfork Y$ on Z_H iff a sequence of obstructions $O_n(f,K) \in H^n(X^H/_{N(H)}, X_H/_{N(H)}, \pi_{n-1}(V_{(H)}))$ vanishes. Here $V(H)$ is a function of the components of X^H. The value of $V(H)$ at a component P of X^H is

$$V(K)_x = \mathrm{Hom}^S_H(\nu(N^H,N)_x, \nu(Y,M)_{H,f(x)})$$

for $x \in P \subset X^H$.

Moreover let $\hat{H}$ be the set of irreducible representations of G,

$$\nu(N^H,N)_x = \Sigma_{\chi\in\hat{H}} a_\chi \chi, \quad \nu(Y,M)_{H,f(x)} = \Sigma_{\chi\in\hat{H}} b_\chi \chi$$

where a_χ and b_χ are integers, $x \in C$.

$D_\chi = \mathrm{Hom}_H(\chi,\chi)$ is a division algebra, $\dim D_\chi \equiv d_\chi$. Then

$V(H)_x = \Pi_{\chi\in\hat{H}} GL(a_\chi, D_\chi)/GL(a_\chi - b_\chi, D_\chi)$.

REMARK (i) $\dim X^H = 0$ or $\dim Y^H - \dim M^H + \dim N^H$

(ii) The cohomology obstruction classes $O_*(f,K)$ should be understood in two ways; as components of X^H if $X^H = \bigcup_1^n X^H_j$, $x_j \in X^H_j$, $X^J_H = X_H \cap X_j$ then $O_*(f,H) = \prod_{j=1}^n H^*(X^H_j/_{N(H)}, X^j_H/_{N(H)}, \pi_{*-1}V(H)_{x_j})$ and as representations of G

$$O_*(f,H) = \pi_{\chi\in\hat{H}} O_*(f)_\chi$$

$$O_*(f)_\chi \in H^*(X^H/_{N(H)}, X_H/_{N(H)}, \pi_{*-1}V(H)^\chi)$$

(iii) If $\dim X^H - \min_{\substack{\chi\in\hat{H}\\ b_\chi\neq 0}} \{d_\chi(a_\chi - b_\chi + 1) - 1\}$

then the obstruction $O_n(H,f) = 0$ for all n.

Recall that ψ is a cross section of the fibration

$$F = C \times_{\mathcal{G}} \Omega^2_-(\mathcal{G}_E) \longrightarrow B = C/\mathcal{G}$$

which is a smooth associated vector bundle of the principal bundle $\hat{C} \to \hat{B}$. Let Z be the zero section $M_1 = \{\nabla \in B \mid \psi_1(\nabla) = 0$ is the moduli space of irreducible connections, which is perturbed at the reducible self-dual connections.

Let $X = M_1^G$, $X_0 = \{M_{\lambda_0} \cup$ open cone neighborhoods at each self-dual reducible connections$\} \cap X$,

then $X \setminus X_0$ is compact.

Now we would like to apply Theorem 6.11. In our case $H = G = Z_2$, $Z_{h-1} = \phi$, $X_H = \phi$, $X = X^H$ and by the construction of M_1^G, the map $\psi_1 \colon B \to F$ has a restriction ψ_1^H such that $\psi_1^H \pitchfork Z^H$ throughout X. Let us consider the obstruction classes $O_n(\psi) \in H^n(X, X_0; \pi_{n-1}(V(H)))$ where $V(H)$ is a fiber bundle over X. The fiber over $x \in X$ is $V(H)_x = \mathrm{Hom}^s_H(\nu(B^H, B)_x, \nu(Z,F)_{H,x})$ where x is an irreducible G-invariant self-dual connection. From the local structure at $x = \nabla$, $T_\nabla B = \mathrm{Ker}\, \delta^\nabla = R^{k+5} \oplus V_1$ for some k. By (6.9), the map $\psi_1 \colon R^{k+5} \oplus V_1 \to R^k \oplus W_1$ is splitted as following. For any $\nabla \in X \setminus X_0$ and $h(\nabla) = g(\nabla)$ if $(hg)^2 = +1$, then $\psi = (Q, d_-) \colon (R_+^{k_1+1} \oplus R_-^{k_2+4}) \oplus V_1 \to (R_+^{k_1} \oplus R_-^{k_2}) \oplus W_1$, if $(hg)^2 = -1$ $\psi = (Q, d_-^\nabla) \colon (R_+^{k_1+3} \oplus R_-^{k_2+2}) \oplus V_1 \to (R_+^{k_1} \oplus R_-^{k_2}) \oplus W_1$. Here the sign $\pm$

means the ± 1 eigenspace of h. If $(hg)^2 = 1$, then $\nu(B^H,B)_x = R_-^{k_2+4} + (V_1)_-$

$$\nu(Z,F)_x = \Omega_-^2(\mathcal{G}_E) = R^k \oplus W_1 = (R_+^{k_1} \oplus (W_1)_+) \oplus (R_-^{k_2} \oplus (W_1)_-)$$

$$\nu(Z,F)_{H,x} = (R_-^{k_2} \oplus (W_1)_-)$$

where (V_1 and W_1), (V_{1+} and W_{1+}) and (V_- and V_{1-}) are G-equivariant Hilbert space isomorphisms by d_-^∇. Thus the fiber

$$V(H)_x = \mathrm{Hom}_H^s(\nu(B^H,B)_x,\ \nu(X,F)_{H,x})$$

$$= \mathrm{Hom}^s(R_-^{k_2+4} \oplus (V_1)_-,\ R_-^{k_2} \oplus (W_1)_-)$$

(6.12)
$$= \begin{cases} \text{contractible if } \dim V_{1^-} = \infty \\ V_{k_2+4,k_2} \quad \text{if } \dim V_{1^-} < \infty \end{cases}$$

If $(hg)^2 = -1$, then $\nu(B^H,B)_x = R_-^{k_2+2} \oplus (V_1)_-$

$$\nu(Z,F)_{H,x} = R_-^{k_2} \oplus (W_1)_-$$

(6.13) The fiber $V(H)_x = \mathrm{Hom}^s(R_-^{k_2+2}, R_-^{k_2})$ is the Stiefel manifold V_{k_2+2,k_2}, because we only consider a compact perturbation.

THEOREM 6.14 [28]. The Stiefel manifold $V_{n,k}$ is arcwise-connected and

$$\pi_i(V_{n,k}) = 0 \quad \text{if} \quad i < n - k$$

$$\pi_{n-k}(V_{n,k}) = \begin{cases} \text{infinite cyclic group}, & \text{if } n-k \text{ is even or } k=1 \\ Z_2, & \text{if } n-k \text{ is odd and } k > 1. \end{cases}$$

By (6.12), (6.13), and (6.14) we have

THEOREM 6.15. In the bundle $V(H) \to X$, the fiber has the fundamental groups as following.

$$\pi_i(V(H)_x) = \begin{cases} Z;\ i = 2, & \text{if } (hg)^2 = -1 \\ 0;\ i \leq 3, & \text{if } (hg)^2 = +1 \end{cases}$$

where $h(x) = h(\nabla) = g(\nabla)$.

Moreover if $(hg)^2 = 1$, then the obstructions cohomology class $O_n(\psi) \in H^n(X,X_0;\pi_{n-1}(V(H))) \equiv 0$ for all n.

However the compact set $X \subset M_1^{\hat{}H} = M_1^{\hat{}G} \subset M_1$, and $\dim M_1 = 5$. This is incorrect because $\hat{M}_1$ may not be a manifold. By corollary 4.14 $M_1^{\hat{}G}$ is a disjoint union of 1-dimensional manifold components and 3-dimensional manifold components which are corresponded by $h(\nabla) = g(\nabla)$ $(hg)^2 = 1$ or $(hg)^2 = -1$ respectively. Thus $X = \bigcup_i X_i^1 \bigcup_i X_i^3$ where dim $X_i^1 = 1$, dim $X_i^3 = 3$. If $h(\nabla) = g(\nabla)$, $(hg)^2 = -1$, then the obstruction cohomology classes $\mathcal{O}_3(\psi) \in H^3(X_i^3, X_{i0}^3; Z)$ where $X_{i0}^3 = X_i^3 \cap X_0$.

THEOREM (6.16). To perturb $\psi: B \to C \times_{\mathfrak{g}} \Omega_-^2(\mathfrak{g}_E)$ G-transversal throughout a neighborhood of M^G there are the obstructions $\mathcal{O}_3(\psi) \in H^3(X, X_0; Z)$.

If the obstructions $\mathcal{O}_3(\psi) = 0$, then the G-section ψ has a smooth compact G-perturbation $R_- + \sigma$ of the self-dual Yang-Mills equations which is transversal to the zero section throughout a small neighborhood of M^G.

CHAPTER VII

PERTURBATION OF METRIC ON M SO THAT A NEIGHBORHOOD OF M^G IS A MANIFOLD

As in the Chapter V, let $C = C^k(GL(TM))$ be the set of C^k-automorphisms of the tangent bundle, G is the group Z/2, C^G is the G-fixed point set of C. For a large fixed K, we define a map

$$\Phi: \hat{C}_{\ell-1} \times C^G \longrightarrow \Omega^2_-(\mathcal{G}_E)_{\ell-2} \text{ by}$$

$$\Phi(\nabla,\phi) = P_-(\phi^{-1*}R^\nabla).$$

Let $\pi: \hat{C} \to \hat{B} = \hat{C}/\mathcal{G}$ be the projection. $M^0 = M \setminus M^G$ is an open dense subset of M.

LEMMA 7.1. <u>For each</u> $x \in M^0$ <u>there is an open neighborhood</u> U <u>of</u> x <u>such that for each</u> $\sigma \in T(C)$ <u>there exist a</u> $\tau \in T(C^G)$ <u>such that</u> $\sigma|U = \tau|U$.

PROOF. For each $x \in M^0$ choose a neighborhood U of x such that $h(U) \cap U = \phi$ where $\langle h \rangle = G$. Note $\sigma \in T(C) = C^k(END(TM))$ and $T(C^G) = C^k(END(TM)^G)$. Choose a cutoff function $f: M^0 \to [0,1]$ such that

$f|U \equiv 1$ and $h(\text{support } f) \cap (\text{support } f) = \phi$. Let $\bar{\sigma} = f\sigma$, then $h[\sup(\bar{\sigma})] \cap [\sup(\bar{\sigma})] = \phi$. Since $f|U \equiv I$, $\sigma = \bar{\sigma}$ on U. By invariantization $\tau \equiv \sum_{h\in G} h^*(\bar{\sigma})$, $h(\tau) = \tau$. We have $\tau \in T(C^G)$ such that $\tau = \sigma$ on U, because for any $y \in U$

$$\tau(y) = \sum_{h\in G} h^*(\bar{\sigma})(y) = \sum_{h\in G} h\cdot\bar{\sigma}(h^{-1}(y))$$

$$= \bar{\sigma}(y) = \sigma(y).$$

Note that this Lemma 7.1 is true for any finite group G. The following theorem is one of our main theorem in this section. To prove this theorem we will follow [[14], Thm 3.4]. However in our case the zero may not be a regular value of Φ because we replace C by C^G. So we should restrict the domain of Φ to a suitably chosen open subset of $\hat{C} \times C^G$.

LEMMA 7.2. ([14]). <u>Suppose</u> $R \in \Lambda_+^2 V^* \otimes W$ <u>and</u> $\phi \in \Lambda_-^2 V^* \otimes W$ <u>satisfies</u> $(r^*R, \phi) = 0$ <u>for all</u> $r \in g\ell(V)$. <u>Then the images</u> $\mathrm{Im}(R)$ <u>and</u> $\mathrm{Im}(\phi)$ <u>are orthogonal</u>.

THEOREM 7.3. <u>There is an open G-set</u> $\mathcal{O}$ <u>of</u> $\hat{C} \times C^G$ <u>such that</u>

(i) <u>the restriction map</u> $\Phi: \mathcal{O} \to \Omega_-^2(\mathfrak{g}_E)$ <u>is smooth and has the zero as a regular value.</u>

(ii) <u>If</u> $\Phi(\nabla,\psi) = 0$, $\nabla \in \hat{M}^G$ <u>then</u> $\pi^{-1}(\hat{M}^G) \times \{\psi\} \subset \mathcal{O}$, <u>where</u> $\pi: \hat{C} \to \hat{B}$ <u>is the projection</u>. $\langle h \rangle = G = Z_2$.

PROOF. It is sufficient to prove the differential $\delta\Phi_{(\nabla,\psi)}$ surjective whenever $\Phi(\nabla,\psi) = 0$ for $(\nabla,\psi) \in A \times C^G$ where $A = \pi^{-1}(M^{\wedge G})$. First if $\Phi(\nabla,\psi) = 0$ and $h(\nabla) = \nabla$, then the differential map

$$\delta\Phi_{(\nabla,\psi)}: \Omega^1(\mathcal{G}_E) \times C^k(\text{END TM})^G \longrightarrow \Omega^2_-(\mathcal{G}_E)$$

splits into two pieces

$$\delta\Phi_{(\nabla,\psi)} = \delta_1\Phi_{(\nabla)} + \delta_2\Phi_{(\psi)}: \Omega^1(\mathcal{G}_E) \oplus C^k(\text{END TM})^G \longrightarrow \Omega^2_-(\mathcal{G}_E)$$

where $(\delta_1\Phi_{(\nabla)})(A) = P_-(\psi^{-1*}DA)$, $(\delta_2\Phi_\psi)(r) = P_-(\psi^{-1*}(r^*R^\nabla))$ for $A \in \Omega^1(\mathcal{G}_E)$ and $r \in C^k(\text{END TM})^G$. Show that $\text{coker}(\delta\Phi) = 0$. If $\phi \in \text{coker}(\delta\Phi)$, then $\phi \in \text{coker}(\delta_1\Phi)$ so that

$$0 = \int_M (P_-\psi^{-1*}DA,\phi)_g = \int_M(\nabla A,\psi^*(\phi))_{\psi^*(g)} = \int_M(A,\nabla^*\tilde{\phi})_{\psi^*(g)}$$

for all $A \in \Omega^1(\mathcal{G}_E)$ where $\tilde{\phi} = \psi^*(\phi)$. Since ϕ is continuous we have the pointwise equation $D^*\tilde{\phi} = 0$. Since $\phi \in \text{coker}(\delta_2\Phi)$,

$$0 = \int_M(P_-\psi^{-1*}(r^*R^\nabla),\phi)_g = \int_M(r^*R^\nabla,\tilde{\phi})_{\psi^*(g)}$$

for all $r \in C^k(\text{END TM}))^G$.

Since $h(D) = D$, $(r^*R^\nabla,\tilde{\phi})_{\psi^*(g)} = (r^*R^\nabla,h\tilde{\phi})_{\psi^*(g)}$. So $0 = \int_M(r^*R^\nabla,\tilde{\phi} + h\tilde{\phi})_{\psi^*(g)}$ for all $r \in C^k(\text{END}(\text{TM})^G)$.

Thus we have $(r^*R^\nabla,\tilde\phi + h\tilde\phi)_{\psi^*(g)} = 0$ at each point of M^0. Since $2\tilde\phi = (\tilde\phi + h\tilde\phi) + (\tilde\phi - h\tilde\phi)$

$$\begin{aligned}(r^*R^\nabla,\tilde\phi - h\tilde\phi)_{\psi^*(g)} &= (r^*R^\nabla,h(\tilde\phi - h\tilde\phi))_{\psi^*(g)} \\ &= -(r^*R^\nabla,\tilde\phi - h\tilde\phi)_{\psi^*(g)}\end{aligned}$$

Thus we have $(r^*R^\nabla,\tilde\phi - h\tilde\phi)_{\psi^*(g)} = 0$. By adding $0 = (r^*R^\nabla,2\tilde\phi)_{\psi^*(g)}$, so $(r^*R^\nabla,\tilde\phi)_{\psi^*(g)} = 0$ at each point $x \in M^0$.

Second if $\Phi(\nabla,\psi) = 0$ and $h(D) = g(D)$ for some gauge transformation $g \in \mathcal{J}$, then $\sigma^2 = \pm 1$ and $\sigma(\nabla) = \nabla$ where $\sigma = hg$. Again since $\phi \in \operatorname{coker} \delta_2\Phi$, $0 = \int_M (r^*R^\nabla,\tilde\phi)_{\psi^*(g)}$ where $\tilde\phi = \psi^*(\phi)$ and for all $r \in C^k(\mathrm{END}\ TM)^G$ since $\sigma^2\tilde\phi = \tilde\phi$, $\tilde\phi + \sigma\tilde\phi$ is σ-invariant and h-invariant in M

$$(r^*R^\nabla,\tilde\phi)_{\psi^*(g)} = (\sigma r^*R^\nabla,\sigma\tilde\phi)_{\psi^*(g)} = (r^*R^\nabla,\sigma\tilde\phi)_{\psi^*(g)}.$$

Thus we have $0 = \int_M (r^*R^\nabla,\tilde\phi + \sigma\tilde\phi)_{\psi^*(g)}$ and so $(r^*R^\nabla,\tilde\phi + \sigma\tilde\phi)_{\psi^*(g)} = 0$. As above $(r^*R^\nabla,\tilde\phi - \sigma\tilde\phi)_{\psi^*(g)} = 0$.

By adding we have $(r^*R^\nabla,\tilde\phi)_{\psi^*(g)} = 0$ at each point $x \in M^0$.

By Lemma 7.1 and 7.2, the images $\mathrm{Im}(R^\nabla)$ and $\mathrm{Im}(\tilde\phi)$ are pointwise orthogonal on M^0 and so on M. At each point which R^∇ and $\tilde\phi$ are non-zero, one of R^∇ and $\tilde\phi$ has rank 1. Sketch the rest of the proof.

Since R^∇ is self-dual and $\tilde{\phi}$ is anti-self-dual with respect to the metric $\psi^*(g)$, $\nabla R^\nabla = \nabla^* R^\nabla = \nabla\phi = \nabla^*\tilde{\phi} = 0$. Suppose $\tilde{\phi} \not\equiv 0$. Let $\tilde{\phi} \neq 0$ and R^∇ have rank 2 in some neighborhood of a point. Write $\tilde{\phi} = a \otimes u$ where $a \in \Omega^2_-$, $u \in \mathfrak{g}_E$ with $|u| = 1$. Then we have $\nabla u = 0$. By Lemma 7.2 $(R^\nabla, u) = 0$ also $\nabla^2 u = [R^\nabla, u] = 0$. This cannot happen for non zero vectors u and R^∇ on R^3. So R^∇ has at most one-dimension. Suppose $F = \sigma \otimes u$ with $|u| = 1$ some open set. Then $\nabla u = 0$, and the complement of $\{F \equiv 0\}$ is connected otherwise $\nabla^*\nabla + \nabla\nabla^*$ has negative eigenvalues on a domain. Then we can extend u on M such that $\nabla u = 0$. But ∇ is irreducible since $k = 1$, $F \not\equiv 0$. Thus we prove $\tilde{\phi} \equiv 0$ and so $\phi \equiv 0$.

Construct an open set $\mathcal{O}$ of $\mathcal{C}^\wedge \times \mathcal{C}^G$ as follows:

$$U_1 \equiv \bigcup \{U_\nabla \times V_\phi \mid \Phi(\nabla,\phi) = 0, h\nabla = \nabla, U : G\text{-set}, \delta\Phi \text{ is onto on } U_\nabla \times V_\psi\}$$

$$U_2 \equiv \bigcup \{U_\nabla \cup U_{g(\nabla)} \times V_\phi \mid \Phi(\nabla,\phi) = 0, h(\nabla) = g(\nabla), U_\nabla \cup U_{g(\nabla)} : G\text{-set}, \delta\Phi \text{ is onto}\}$$

Set $\mathcal{O} = U_1 \cup U_2$. Then by construction $\mathcal{O}$ is open G-set and the restriction map $\Phi: \mathcal{O} \to \mathcal{C}^\wedge \times \mathcal{C}^G \to \Omega^2_-(\mathfrak{g}_E)$ has zero as a regular value.

We consider the following diagram

$$\begin{array}{ccccccc}
\Phi^{-1}(0) & \hookrightarrow & \mathcal{O} & \hookrightarrow & \mathcal{C}^\wedge \times \mathcal{C}^G & \xrightarrow{\Phi} & \Omega^2_-(\mathfrak{g}_E) \\
\downarrow \pi & & \downarrow \pi & & \downarrow \pi \times \text{id} & & \\
\Phi^{-1}(0)/\mathcal{G} & \longrightarrow & \mathcal{O}/\mathcal{G} & \longrightarrow & \mathcal{B}^\wedge \times \mathcal{C}^G & & \\
\downarrow \bar{\pi} & & \downarrow & & \downarrow & & \\
\mathcal{C}^G & = & \mathcal{C}^G & = & \mathcal{C}^G & &
\end{array}$$

The set $\phi^{-1}(0)$ is a manifold in $\mathcal{O}$. For each metric $\psi \in C^G$ the fixed point set $M_\psi^{\hat{}G}$ in the moduli space $M_\psi^{\hat{}}$ is contained in $\phi^{-1}(0)/\mathcal{G}$ because $M_\psi^{\hat{}G} = \{(\nabla,\psi) \in C^{\hat{}} \times \{\psi\} | \phi(\nabla,\psi) = 0, \ (\nabla) = \nabla$ or $h(\nabla) = g(\nabla)\ /\mathcal{G}$. This completes the proof of the Theorem 7.3.

LEMMA 7.4. $\phi^{-1}(0)/\mathcal{G} \subset \mathcal{O}/\mathcal{G}$ is a manifold.

PROOF. First clearly $\mathcal{O}/\mathcal{G}$ is a manifold by the slice theorem.

For any $(\nabla,\phi) \in \phi^{-1}(0) \subset \mathcal{O}$ we have a neighborhood $U_\nabla \times V_\phi \subset C^{\hat{}} \times C^G$. Since $TC^{\hat{}} = \Omega^1(\mathcal{G}_E) = \mathrm{Im}\nabla \oplus \mathrm{Ker}\,\nabla^*$ and $\delta_1\Phi|_{\mathrm{Im}\nabla} = 0$, the differential $\delta\bar{\Phi}\colon \mathcal{O}/\mathcal{G} \to \Omega^2_-(\mathcal{G}_E)$ has the same image of $\delta\Phi$ on $\mathcal{O} \to \Omega^2_-(\mathcal{G}_E)$. Thus the induced map $\bar{\Phi}\colon \mathcal{O}/\mathcal{G} \to \Omega^2_-(\mathcal{G}_E)$ has zero as a regular value. We complete the proof.

LEMMA 7.5. The projection $\phi^{-1}(0)/\mathcal{G} \xrightarrow{\pi} C^G$ is a Fredholm map.

PROOF. We consider the construction of $\mathcal{O} \subset C^{\hat{}} \times C^G$. The differential $\delta\bar{\pi}\colon T_{(\nabla,\phi)}(\phi^{-1}(0)/\mathcal{G}) = \{(A,r) \in \Omega^1(\mathcal{G}_E) \times TC^G | \delta_1\phi(A) + \delta_2\phi(r) = 0$ and $\nabla^*A = 0\} \to C^k(\mathrm{END}(TM))^G$. Since $\mathrm{Ker}\,\delta\bar{\pi} = \{(A,r)\colon \delta_1\phi(A) = \nabla^*A = r = 0\} = H^1_\nabla$ and $\mathrm{Im}\,\delta\bar{\pi} = (\delta_2\phi)^{-1}[\mathrm{Im}\delta_1\phi|_{\mathrm{Ker}\nabla^*}] = (\delta_2\phi)^{-1}[\mathrm{Im}\,\delta_1\phi]$. We have coker $\delta\bar{\pi} \approx H^2_\nabla$ because $\delta\Phi_{(\nabla,\phi)}$ is onto. Since ∇ is irreducible self-dual, Ind $\bar{\pi}$ = index of the fundamental elliptic complex for $\nabla = H^1_\nabla - H^2_\nabla$ has a numerical index 5.

Now we use the Sard-Smale Theorem for the Fredholm map $\bar{\pi}\colon \phi^{-1}(0)/\mathcal{G} \to C^G$, between paracompact Banach manifolds. There is the set of regular values of $\bar{\pi}$ which is an open dense set in C^G, because $\dim(H^2)$ is an upper semi-continuous integer valued function on $\phi^{-1}(0)/\mathcal{G}$. If ϕ is a regular value, then $\bar{\pi}^{-1}(\phi)$ is a manifold with dimension 5, which is a neighborhood of $\hat{M}_\phi^G$ in $\hat{M}_\phi$ with respect to the G-invariant metric $\phi^*(g)$ on M.

THEOREM 7.6. There is an open dense set in C^G such that the moduli space $\hat{M}$ of irrational connections is a manifold in a G neighborhood of $\hat{M}^G$ for each metric in the dense set.

Let ∇ be a reducible self-dual G-invariant connection in M. We have a bundle splitting $E = \ell \oplus \bar{\ell}$ and so $\mathcal{G}_E = R \oplus \eta$. Also we can split the complex:

$$0 \longrightarrow \Omega^0(\mathcal{G}_E) \xrightarrow{\nabla} \Omega^1(\mathcal{G}_E) \xrightarrow{\nabla} \Omega^2_-(\mathcal{G}_E) \longrightarrow 0$$

$$\equiv (0 \longrightarrow \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d_-} \Omega^2_- \longrightarrow 0) \oplus (0 \longrightarrow \Omega^0(\eta) \xrightarrow{D} \Omega^1(\eta) \xrightarrow{D_-} \Omega^2_-(\eta) \longrightarrow 0)$$

As Theorem 7.2 the main differences between [14] and our case are that (i) We should consider separately $h(D) = g(D), (hg)^2 = 1$ and $(hg)^2 = -1$ where g act on η with weight 2.

To prove the map $Q: \Omega^1(\eta)\backslash 0 \times C^G \to \Omega^0(\eta) \oplus \Omega^2_-(\eta)$ given by $(A,\phi) \mapsto (D^*\phi^{-1*}A, P_-\phi^{-1*}DA)$ is a submersion throughout $Q^{-1}(0)$, we should use the condition (i) as Theorem 7.2 because we restrict to the G-invariant metrics C^G. Then $Q^{-1}(0)$ is a manifold. The projection $\pi: Q^{-1}(0) \to C^G$ has index 6 by considering the splitted complex. Again using the Sard-Smale Theorem and the upper continuity of $\dim H_D^2$, we have

THEOREM 7.7. <u>There is an open dense set in</u> C^G <u>such that the second cohomology group</u> $H_D^2 = 0$ <u>in the fundamental complex of each</u> G-<u>invariant reducible connection</u> ∇ <u>in</u> M.

Since the finite intersection of open dense sets in C^G is also an open and dense set in C^G. Finally by the Theorem 7.6 and 7.7, we have

THEOREM 7.8. <u>There is a dense set in the set</u> C^G <u>of</u> G-<u>invariant metrics on</u> M <u>such that the moduli space</u> M <u>is a manifold in a</u> G-<u>neighborhood of the fixed point set</u> M^G <u>for each metric in the dense set. Moreover for these metrics Petrie obstruction classes vanish.</u>

REMARK. In Theorem 7.2 if we do not restrict the map $\Phi: \hat{C} \times C^G \to \Omega_-^2(\mathcal{J}_E)$, then the zero may not be a regular value. In Theorem 7.7, if we do not choose a G-invariant reducible connection, the map Q may not be a submersion. So we need G-equivariant compact perturbation at the free part to get G-manifold M. More generally if G is a finite cyclic group of order 2^n, then we can follow Theorem 7.2 by considering the order 2^n and in case $h(\nabla) = g(\nabla)$ the element (hg) has again order 2^m for some $m \leq n$.

THEOREM 7.9. <u>If</u> G <u>is a finite cyclic group of order</u> 2^n, <u>then there is a dense set in the set</u> C^G <u>of</u> G-<u>invariant metrics on</u> M, <u>such that the moduli space</u> $\mathcal{M}$ <u>is a manifold in a</u> G-<u>neighborhood of the fixed point set</u> $\mathcal{M}^G$ <u>for each metric in the dense set</u>.

SKETCH OF PROOF. As Theorem 7.2, $(r^*R^\nabla, \tilde{\phi} + h\tilde{\phi} + \cdots + h^{2^n-1}\tilde{\phi})_{\psi^*(g)} = 0$ and $(r^*R^\nabla, \tilde{\phi} - h\tilde{\phi} + \cdots + h^{2^{n-1}}\tilde{\phi} - \cdots - h^{2^n-1}\tilde{\phi})_{\psi^*(g)} = 0$. Add and divide by 2, $(r^*R^\nabla, \tilde{\phi} + h^2\tilde{\phi} + \cdots + h^{2^{n-1}}\tilde{\phi} + \cdots) = 0$ continue this process, we have

$$(r^*R^\nabla, \tilde{\phi} + h^{2^{n-1}}\tilde{\phi})_{\psi^*(g)} = 0$$

$$(r^*R^\nabla, \tilde{\phi} - h^{2^{n-1}}\tilde{\phi})_{\psi^*(g)} = 0$$

so we have $(r^*R^\nabla, \tilde{\phi})_{\psi^*(g)} = 0$. If $h(\nabla) = g(\nabla)$, $g \neq \pm 1$ and ∇ is irreducible, then $hg(\nabla) = \nabla, (hg)^{2^n} = hgh^{2^n-1}. h^2gh^{2^n-2}\cdots g = \pm 1$. If $h(\nabla) = g(\nabla)$, $g \notin \Gamma_\nabla$ and ∇ is reducible, then $hg = g_1h$ for some $g_1 \in \Gamma_\nabla$.

CHAPTER VIII

PERTURBATION ON THE FREE PART OF $\mathcal{M}$

In the Chapter VII, we found a G-invariant metric on M such that the moduli space $\mathcal{M}$ is a manifold in a G-neighborhood of the fixed point set $\mathcal{M}^G$. We will fix this G-invariant metric on M. Then the map $B \xrightarrow{\psi} F = C \times_{\mathcal{G}} \Omega^2_-(\mathcal{G}_E)$ is the Fredholm G-map which is transverse to the zero section throughout a G-neighborhood $N(\mathcal{M}^G)$ of the fixed point set $\mathcal{M}^G$. Let $Y = \mathcal{M} \setminus \{N(\mathcal{M}^G) \cup \text{End of } \mathcal{M}\}$.

Then Y is a compact subset of $\mathcal{M}$ and $\mathcal{M}\setminus Y$ is a smooth 5-dimensional manifold with some singular points. For each $\nabla \in Y$ we can choose a local coordinate $\mathcal{O}_{\nabla\cdot\varepsilon} = \{A \in \Omega^1_3(\mathcal{G}_E) \mid \delta^\nabla A = 0, \ \|A\| < \varepsilon\}$ if ∇ is irreducible, otherwise $\mathcal{O}_{\nabla\cdot\varepsilon}/U(1)$ is a local coordinate chart at ∇ such that $h(\mathcal{O}_{\nabla\cdot\varepsilon}) \cap \mathcal{O}_{\nabla\cdot\varepsilon} = \phi$ where h is the generator of $G = Z_2$. To see the local structure of the map $\psi: B = C/\mathcal{G} \to F = C \times_{\mathcal{G}} \Omega^2_-(\mathcal{G}_E)$ and the relation with G-action only, let us consider the following diagram.

$$
\begin{array}{ccc}
V_0 + V_1 = V \supset \mathcal{O}_{\nabla\cdot\varepsilon} & \xrightarrow[(Q_1,h_1)]{\psi} & \Omega^2_-(\mathcal{G}_E)_\nabla = W_0 \oplus W_1 \\
\Big\downarrow h & & \Big\downarrow h \\
V'_0 + V'_1 = h_*(V) \supset h(\mathcal{O}_{\nabla\cdot\varepsilon}) & \xrightarrow[(Q_2,h_1)]{\psi} & \Omega^2_-(\mathcal{G}_E)_{h(\nabla)} = W'_0 \oplus W'_1
\end{array}
$$

Since ψ is a G-map, above diagram commutes. Since ψ is a Fredholm map with index 5, ψ becomes locally $\psi = (Q,L): V_0 \oplus V_1 \to W_0 \oplus W_1$ by some G-equivariant diffeomorphism where Q,L are G-maps, $d\psi|_{V_1} \doteq L|_{V_1}: V_1 \to W_1$ is a Hilbert space isomorphism, $Q|_{V_0}: V_0 = R^{k+5}(C^{k+3}) \to W_0 \equiv R^k(C^k)$ is also G-map with $dQ = 0$. (if ∇ is reducible) (c.f. F.U. Lemma 4.7). Since h is diffeomorphic, we can locally identify h with its differential at the origin ∇. Let $d\psi_\nabla = L_1$, $d\psi_{h(\nabla)} = L_2$. Since $\psi h = h\psi$ we get $L_2 h = hL_1$. Since V_0 is the kernel of L_1 and $L_1: V_1 \to W_1$ is an isomorphism

$$L_2[hV_0] = h[L_1V_0] = h(0) = 0, \quad \text{so} \quad h(V_0) \in \text{Ker } L_2$$

$$L_2[hV_1] = h[L_1V_1] = h(W_1), \quad \text{so} \quad L_2: h(V_0) \overset{\text{iso}}{\cong} h(W_1).$$

Thus we have the canonical splitting $\psi = (Q_2,L_2): h(V_0) \oplus h(V_1) \to h(W_0) \oplus h(W_1)$ at a neighborhood of $h(\nabla)$.

LEMMA 8.1. <u>For each</u> $\nabla \in Y$ <u>the generator</u> $h \in Z_2$ <u>preserves the local splitting of the Fredholm map</u> $\psi: B \to F$.

With these preliminaries let us perturb ψ on $Y \subset M$. Suppose that $\nabla \in Y$ is reducible. Then $h(\nabla)$ is also reducible. Using our usual compact G-invariant perturbation $\sigma: \mathcal{O}_{\nabla\cdot\varepsilon} \to \Omega^2_-(\mathcal{J}_E)$.

We have a new section $\psi_1 = \psi + \sigma: B \to F$ which transverse to the zero section throughout $\mathcal{O}_{\nabla,\varepsilon}$. Define a perturbation on $h(\mathcal{O}_{\nabla,\varepsilon})$ by $\sigma(h(A)) = h\sigma(A)$. We have a G-equivariant section $\psi_2 = \psi_1 + \sigma: B \to F$ which transverse to the zero section throughout $\mathcal{O}_{\nabla\ \varepsilon} \cup h(\mathcal{O}_{\nabla}\ )$. Since this is a compact perturbation if we mod out the zero set at ∇ by $U(1)$, then this reducible connection has a neighborhood which is a cone on Cp^2. Adding such a perturbation at each reducible connection in Y, we have a section $\psi_3: B \to F$ which transverses to the zero section near the reducible connections in $M_1 = \{\nabla \in B | \psi_3(\nabla) = 0\}$. Thus we have.

LEMMA 8.2. <u>Suppose that</u> $\nabla \in Y$ <u>is reducible. Then we have a</u> G-<u>equivariant compact perturbation of</u> ψ <u>so that</u> M_1 <u>has a cone-neighborhood on</u> Cp^2 <u>at</u> ∇.

Let $Y_1 = M_1 - \{N(M^G) \cup \text{End of } M \cup [\text{cones on } Cp^2 \text{ at reducible self-dual connections}]\}$.

Let $\nabla \in Y_1$ be irreducible. Y_1 is compact. The Frehold map ψ_3 is locally splitted; $\psi_3 = (Q,L): \mathcal{O}_{\nabla,\varepsilon} \subset V = V_0 \oplus V_1 \to \Omega^2_-(\mathcal{G}_E) = W_0 \oplus W_1$, where $L = d\psi_3: V_1 \to W_1$ is a Hilbert isomorphism, $V_0 = R^{k+5}$, $W_0 = R^k$, and every map is G-map and every space is G-space.

Choose a smooth cutoff function $\rho \in C_0(\mathcal{O}_{\nabla,\varepsilon})$ and consider the family of perturbations $\sigma_w \equiv \rho \cdot w: \mathcal{O}_{\nabla,\varepsilon} \to R^k_\nabla \subset \Omega^2_-(\mathcal{G}_E)$ for each $w \in R^k = W_0$. As above extend the perturbation by $h(\sigma_w A) = \sigma_{hw}(hA)$ on $h(\mathcal{O}_{\nabla,\varepsilon}) \xrightarrow{\sigma_{h,w}} R^k_{h(\nabla)} \subset \Omega^2_-(\mathcal{G}_E)_{h(\nabla)}$ for each $hw \in R^k_{h(\nabla)}$ (cf. Lemma 8.1).

Considering the G-map $Q_\nabla | R_\nabla^{k+5}: R_\nabla^{k+5} \to R_\nabla^k$ we have a immediate consequence

LEMMA 8.3. $w \in R_\nabla^k$ <u>is a regular value of</u> $Q_\nabla: R_\nabla^{k+5} \to R_\nabla^k$ <u>if and only if</u> $h(w) \in R_{h(\nabla)}^k$ <u>is a regular value of</u> $Q_{h(\nabla)}: R_{h(\nabla)}^{k+4} \quad R_{h(\nabla)}^k$.

We can cover the compact set Y_1 with the supports of a finite numbers of such perturbations. We get a family of perturbations $\psi_w = \psi_3 + \sigma_w + \sigma_{hw_1} + \cdots\cdots + \sigma_{w_n} + \sigma_{hw_n}$ for each $w = (w_1, \cdots\cdots, w_n) \in R^{k_1} \times \cdots\cdots\cdots \times R^{k_n} \equiv R^m$.

We may assume that the supports of the perturbations lie in a small neighborhood of Y_1. Let a smooth mapping $\tilde{\psi}: B \to B^m(\eta) \to F$ be defined by $\tilde{\psi}(x,w) = \psi_w(x)$, where $B^m(\eta) = \{w \quad R^m: \|w\| < \eta\}$.

LEMMA 8.4. <u>For small</u> $\eta > 0$, <u>this mapping</u> $\tilde{\psi}: B \times B^m(\eta) \to F$ <u>is transversal to the zero section</u> $Z \subset F$.

PROOF. Suppose that $(x,w) \in B \times B^m(\eta)$ such that $\tilde{\psi}(x,w) = 0$

(i) if $x \notin$ support of ρ_i for all i, then $\tilde{\psi}(x,w) = \psi_3(x) = 0, \tilde{\psi}$ is transversal already by our constructions.

(ii) if $x \in$ support of ρ_i for some i, then $x \in \operatorname{supp} \rho_i \subset \mathcal{O}_{\nabla_i, \varepsilon}$.

Write $\tilde{\psi}(x,w) = \psi_3(x) + \sigma(x,\bar{w}_i) + \rho_i(x) w_i$. $\bar{w}_i = (w_1 \cdots \hat{w_i} \cdots w_n)$ where $\sigma(x,\bar{w}_i) = \sum_{i \neq j} \sigma_{w_i}(x)$ is uniformly C^1-small.

This is guaranteed by choosing η small after covering with a finite number of coordinate charts.

$d(\psi_3 + \sigma)_x: V = V_0 \oplus V_1 \to W = W_0 \oplus W_1$ will still be transverse to W_1. Also σ_i is the map $R^{k+5} \times R^{k_i} \xrightarrow{\rho_i \times id} R \times R^{k_i} \xrightarrow{\text{scalar multi}} R^{k_i}$ which has clearly surjective differential. Namely the w_i-spaces carried onto w_0.

Hence the total differential is surjective. i.e. $\psi \pitchfork Z$.

By Sard's theorem for families, the map $\psi_w = \psi_3 + \sigma_{w_1} + \sigma_{hw_1} + \cdots + \sigma_{w_n} + \sigma_{hw_n}$ is transversal to the zero section for almost all $w \in B^m(\eta)$.

LEMMA 8.5. $\psi_w: B \to F$ <u>is a G-map</u>.

PROOF. If $A \notin \operatorname{supp} \rho_i$ for all i, then $h(A) \notin \operatorname{supp} \rho_i$ for all i and $\psi_w(hA) = \psi_3(hA) = h\psi_3(A) = h\psi_w(A)$. If $A \in \operatorname{supp} \rho_i$ for some i, then $A \in \operatorname{supp} \rho_i \subset \mathcal{O}_{\nabla\cdot\varepsilon}$ and $h(A) \in h(\mathcal{O}_{\nabla\cdot\varepsilon})$. By our construction $\mathcal{O}_{\nabla\cdot\varepsilon} \cap h(\mathcal{O}_{\nabla\cdot\varepsilon}) = \emptyset$

$$\begin{aligned}\psi_w(hA) &= \psi_3(hA) + \sigma_{w_1}(hA) + \sigma_{hw_1}(hA) + \cdots + \sigma_{w_n}(hA) + \sigma_{hw_n}(hA) \\ &= h\psi_3(A) + h\sigma_{hw_1}(A) + h\sigma_{w_1}(A) + \cdots + h\sigma_{hw_0}(A) + h\sigma_{w_n}(A) \\ &= h[\psi_3(A) + \sigma_{hw_1}(A) + \sigma_{w_1}(A) + \cdots + \sigma_{hw_1}(A) + \sigma_{w_n}(A)] \\ &= h\psi_w(A).\end{aligned}$$

THEOREM 8.6. There is a compact G-equivariant perturbation $\psi_4 = \psi_3 + \sigma_2$ of the perturbed self-dual equation $\psi_3 = R_- + \sigma_1$ so that the new moduli space $M_2 = \{\nabla \in B\colon \psi_4(\nabla) = 0\}$ is a smooth 5-dimensional G-manifold with λ-singularities, each of which has a neighborhood diffeomorphic to the cone on Cp^2 except the cone point where $\lambda = \text{rank } H^2(M:Z)$.

CHAPTER IX

MORE INDEX COMPUTATION AND REDUCIBLE CONNECTIONS

Let $G = Z_p$ where p is a prime number. Let M be a simply connected, closed, smooth 4-manifold with a positive definite intersection form, and a smooth G-action on it. Let $\pi: E \to M$ be a quaternion line bundle with instanton number one and with G-action on E through bundle isomorphism such that π is a G-map. The moduli space $\mathcal{M}$ of self-dual connections on E is a G-space when we start with G-invariant metric on M. This moduli space $\mathcal{M}$ may not be a manifold. As before we may choose a G-invariant metric on M such that the fixed point set $\hat{\mathcal{M}}^G$ in the space $\hat{\mathcal{M}}$ of irreducible self-dual connections is smooth. In this chapter we will compute the G-index of the fundamental elliptic complex for each G-invariant connection in $\mathcal{M}$ and we will investigate reducible connections.

Assume that the fixed point set F of G-action on M is $F = \{P_i\}_{i=1}^{k_1} \cup \{T^{\lambda_i}\}_{i=1}^{k_2}$ where the P_i's are isolated points and the T^{λ_i}'s are Riemann surfaces with genus λ_i.

Again consider G-invariant elliptic complex (i.e. the self-dual connection ∇ is a fixed point in the moduli space $\mathcal{M}$).

$$0 \longrightarrow \Omega^0(\mathcal{G}_C) \underset{\delta^\nabla}{\overset{d^\nabla}{\rightleftarrows}} \Omega^1(\mathcal{G}_C) \xrightarrow{d_-^\nabla} \Omega^2_-(\mathcal{G}_C) \longrightarrow 0.$$

Assume this complex is complexified. Then as usual we get an equivalent single Dirac operator:

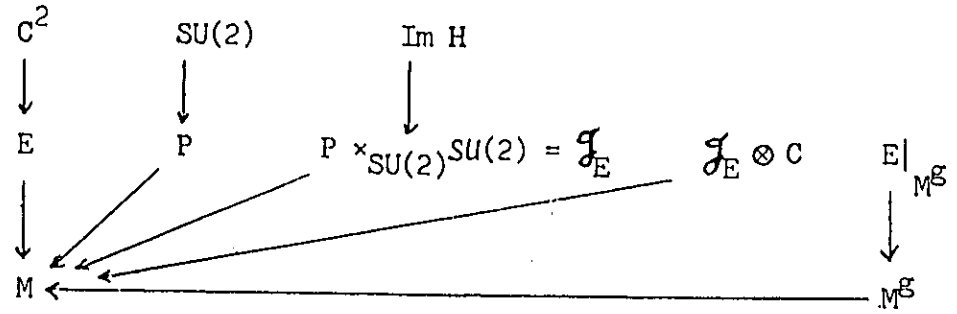
$$D: \Gamma(V_+ \otimes V_- \otimes \mathcal{J}_C) \longrightarrow \Gamma(V_- \otimes V_- \otimes \mathcal{J}_C).$$

Let $K = \{e^{\frac{2\pi i}{p}n} : n = 0,1\cdots\cdots p-1\}$ be the character group of G. In the complex representations which is just the p-th roots of the unity. Let $g = e^{\frac{2\pi i}{p}}$ be a generator of G and let ∇ be an irreducible self-dual G-invariant connection. Then the analytic G-index of D is a virtual representation of G, namely $\mathrm{Index}_G(D) = H^1(\mathcal{J}_C) - H^2(\mathcal{J}_C) \in R(G)$. By Atiyah-Singer Fixed Point Theorem, we can compute the index

$$\mathrm{Index}_g(D) = \mathrm{trace}(g\ \mathrm{index}_G(D)) = (-1)^{\dim M^g} \frac{ch_g(V_+-V_-)ch_g(V_-)(\mathcal{J}_C)td(TM^g \otimes C)}{ch_g(\Lambda_{-1}N^g \otimes C)} \cdot [TM^g]$$

$$= (-1)^{\frac{\dim M^g}{2}} \frac{ch_g(V_+-V_-)ch_g(V_-)ch_g(\mathcal{J}_C)td(TM^g \otimes C)}{e(TM^g)ch_g(\Lambda_{-1}N^g \otimes C)} [M^g]$$

To calculate $ch_g(\mathcal{J}_C)$ let us examine Z_p action on $\mathcal{J}_C$. Consider a diagram



To preserve the Su(2)-structure on $E|_{M^g}$, $g = e^{\frac{2\pi i}{p}}$ acts as

$$\begin{bmatrix} e^{\frac{2 i}{p}k} & 0 \\ 0 & e^{\frac{2\pi i(p-k)}{p}} \end{bmatrix}$$ on $E|_{M^g}$ for some k and so on its associated

principal bundle P. On the associated Lie algebra bundle

$\mathcal{G}_E = P \times_{SU(2)} su(2)$, G acts by conjugation, i.e.

$$\begin{bmatrix} e^{\frac{2\pi ik}{p}} & 0 \\ 0 & e^{\frac{2\pi i(p-k)}{p}} \end{bmatrix} \begin{bmatrix} it & a \\ -\bar{a} & -it \end{bmatrix} \begin{bmatrix} e^{\frac{-2\pi ik}{p}} & 0 \\ 0 & e^{\frac{-2\pi i(p-k)}{p}} \end{bmatrix} = \begin{bmatrix} it & a \\ -\bar{a} & -it \end{bmatrix}$$

So G acts trivially on $\mathcal{G}_E$ and $\mathcal{G}_C = \mathcal{G}_E \otimes_R C$. Thus in our case $ch_g(\mathcal{G}_C) \equiv 3$.

The contribution to the $\mathrm{Index}_g(D)$ ac an isolated fixed point $P_i \in F$.

Let $\theta_1 = \frac{2\pi r_i}{p}$, $\theta_2 = \frac{2\pi s_i}{p}$ represent the representation of g on the normal bundle at P_i in M.

$$\frac{ch_g(V_+ - V_-)ch_g(V_-)}{ch_g(\Lambda_- N^g \times C} = \prod_{t=1}^{2} \frac{e^{\frac{i\theta_t}{2}} - e^{\frac{-i\theta_t}{2}})e^{\frac{-i\theta_t}{2}}}{(1-e^{i\theta_t})(1-e^{-i\theta_t})} \quad (e^{i\theta_1} + e^{i\theta_2})$$

$$= -\frac{1}{2}\left(1 + \cot\frac{\pi r_i}{p}\cot\frac{\pi s_i}{p}\right)$$

Thus we have

$$\text{Iddex}_g(D)\ P_i = -\frac{3}{2}\left(1 + \cot\frac{\pi r_i}{p}\cot\frac{\pi s_i}{p}\right)$$

Next the contribution to the $\text{Index}_g(D)$ on a fixed point component $T^{\lambda_i} \subset F \subset M$ where T^{λ_i} is a Riemann surface with genus λ_i. Let g act on the normal bundle of T^{λ_i} in M by $e^{\frac{2\pi i t_i}{p}}$-multiplication on the fibers.

$$\frac{ch_g(V_+ - V_-)ch_g(V_-)}{e(T^g)ch_f(\Lambda_{-1}N^g\ C)} = \frac{(e^{\frac{x_1}{2}} - e^{\frac{-x_1}{2}})e^{\frac{-x_1}{2}}(e^{\frac{x_2}{2}}e^{\frac{\pi i t_i}{p}} - e^{\frac{-x_2}{2}}e^{\frac{-\pi i t_i}{p}})e^{\frac{-x_2}{2}}e^{\frac{-\pi i t_i}{p}}}{x_1\,(1-e^{x_2}e^{\frac{2\pi i t_i}{p}})(1-e^{-x_2}e^{\frac{-2\pi i t_i}{p}})}\ [e^{x_1} + e^{\frac{2\pi i t_i}{p}}e^{x_2}]$$

$$= \frac{e^{x_1} + e^{x_2}e^{\frac{2\pi i t_i}{p}}}{(1-e^{x_2}e^{\frac{2\pi i t_i}{p}})} = \frac{1+x_1+(1+x_2)e^{\frac{2\pi i t_i}{p}}}{1-(1+x_2)e^{\frac{2\pi i t_i}{p}}} = \frac{(1+e^{\frac{2\pi i t_i}{p}}) + (x_1 + e^{\frac{2\pi i}{p}}x_2)}{(1-e^{\frac{2\pi i t_i}{p}} - e^{\frac{2\pi i t_i}{p}}x_2)}$$

$$= \frac{1}{(1-e^{\frac{2\pi i t_i}{p}})}\ (1+e^{\frac{2\pi i t_i}{p}}) + (x_1 + e^{\frac{2\pi i t_i}{p}}x_2)\ \left[1 + \frac{e^{\frac{2\pi i t_i}{p}}}{1-e^{\frac{2\pi i t_i}{p}}}x_2\right]$$

$$= \frac{1}{(1-e^{\frac{2\pi i t_i}{p}})}\ \left[x_1 + e^{\frac{2\pi i t_i}{p}}x_2 + \frac{(1+e^{\frac{2\pi i t_i}{p}})e^{\frac{2\pi i t_i}{p}}}{(1-e^{\frac{2\pi i t_i}{p}})}x_2\right]$$

$$= \frac{1}{(1-e^{\frac{2\pi i t_i}{p}})}\ x_1 + \frac{2e^{\frac{2\pi i t_i}{p}}}{(1-e^{\frac{2\pi i t_i}{p}})^2}x_2$$

Here we only consider degree one part because, when we evaluate on the fundamental homology class $[T^{\lambda_i}]$, the other parts are all zero. x_1 and x_2 are the Euler classes of the tangent bundle and the normal bundle of T^{λ_i} in M respectively. We can calculate $x_2[T^{\lambda_i}] = m_{T^{\lambda_i}}$ and $x_1(T^{\lambda_i}) = 2 - 2\lambda_i$.

$$\mathrm{Index}_g(G)\Big|_{T^{\lambda_i}} = (-1)\ \frac{ch_g(V_+-V_-)ch_g(V_-)ch_g(\mathcal{J}_C)td(T^g \otimes C)}{e(T^g)\ ch_g(\Lambda_{-1}N^g \otimes C)}[T^{\lambda_i}]$$

$$= -3\left[\frac{1}{\left(1-e^{\frac{2\pi i t_i}{p}}\right)}x_1 + \frac{2e^{\frac{2\pi i t_i}{p}}}{\left(1-e^{\frac{2\pi i t_i}{p}}\right)^2}x_2\right][T^{\lambda_i}]$$

$$= -3\left[\frac{1}{\left(1-e^{\frac{2\pi i t_i}{p}}\right)} + \frac{2e^{\frac{2\pi i t_i}{p}}}{\left(1-e^{\frac{2\pi i t_i}{p}}\right)^2}\ T^{\lambda_i}\right]$$

$$= -6\left[\frac{(1-\lambda_i) + (m_{T^{\lambda_i}} + \lambda_i - 1)e^{\frac{2\pi i t_i}{p}}}{\left(1-e^{\frac{2\pi i t_i}{p}}\right)^2}\right]$$

Hence we have

$$(9.1)\qquad \mathrm{Index}_g(D) = \sum_{i=1}^{k_1} \mathrm{Index}_g(D)\Big|_{P_i} + \sum_{i=1}^{k_2} \mathrm{Index}_g(D)\Big|_{T^{\lambda_i}}$$

$$= \sum_{i=1}^{k_1} \left(-\frac{3}{2}\right) \left(1 + \cot \frac{\pi r_i}{p} \cot \frac{\pi s_i}{p}\right)$$

$$+ \sum_{j=1}^{k_2} (-6) \left[\frac{(1-\lambda_j) + (m_{T^{\lambda_j}} + \lambda_j - 1) e^{\frac{2\pi i t_j}{p}}}{\frac{2\pi i t_j}{p}} \right]$$

For $g^n \in Z_p$ we can calculate the index as (8.2)

THEOREM 9.2.

$$\text{Index}_{g^n}(D) = \sum_{i=1}^{k_1} \left(-\frac{3}{2}\right) \left(1 + \cot \frac{n\pi r_i}{p} \cot \frac{n\pi s_i}{p}\right) + \sum_{j=1}^{k_2} (-t) \left[\frac{(1-\lambda_j) + (m_{T^{\lambda_i}} + \lambda_j - 1) e^{\frac{2n\pi i t_j}{p}}}{\frac{2n\pi i t_j}{p}} \right]$$

where $n = 0, 1, \cdots p-1$ and r_i, s_i, t_i are determined by representations on the normal bundles of the fixed point set in M.

REMARK: (i) In the index calculation, $td(T^g \otimes C) = 1 + \frac{1}{2} c_1(T^g \otimes C) = 1$.

(ii) Above $\text{Index}_g(D)$ is the topological index of D evaluated at g. If we know the exact data, namely the fixed point set, Z_p-representation on the normal bundles and its Euler numbers then by the formula (9.2) we calculate explicitly the topological index.

(iii) For example $G = Z_2$, $F = \{P, s^2\} \subset Cp^2$, $g = -1$. By (8.2)

$$\text{Index}_g(D) = \sum_{i=1}^{k_1} (-\tfrac{3}{2})\ (1 + \cot \frac{\pi r_i}{p} \cot \frac{\pi s_i}{p})$$

$$+ \sum_{j=1}^{k_1} (-6)\ \left[\frac{(1 - \lambda_j) + (m_{\lambda_j}^{T} + \lambda_j - 1)e^{\frac{2\pi i t_j}{p}}}{(1 - e^{\frac{2\pi i t_j}{p}})^2}\right]$$

$$= \sum_{i=1}^{1} (-\tfrac{3}{2})(1 + 0) + \sum_{j=1}^{1} (-6)\ \frac{(1-0) + (1+0-1)(-1)}{[1 - (-1)]^2}$$

$$= -3$$

(iv) For simplicity these topological index

$$\text{Index}_{g^n}(D) \overset{\text{put}}{\equiv} B_n \qquad n = 0,1,\cdots\cdot p-1$$

Again, from the G-invariant fundamental elliptic complex. Let us consider the G action on the virtual representation $H^1_\nabla - H^2_\nabla \in R(G)$ of the cohomology groups. Let us split H^1_∇ and H^2_∇ into the irreducible decompositions

$$H^1_\nabla = H^1_{g^0} \oplus H^1_{g^1} \oplus\cdots\oplus H^1_{g^{p-1}} \quad \text{and}$$

$$H^2_\nabla = H^2_{g^0} \oplus H^2_{g^1} \oplus\cdots\oplus H^2_{g^{p-1}} \quad \text{where} \quad g = e^{\frac{2\pi i}{p}} \in G$$

acts on $H^j_{g^k}$ $(j = 1,2)$ by the complex multiplication $e^{\frac{2\pi i}{p}k}$. Let $\dim_C H^1_\nabla = m$, $\dim_C H^1_{g^i} = m_i$, $\dim_C H^2_\nabla = n$ and $\dim_C H^2_{g^i} = n_i$. For each $g^i \in G$ we have the indices

$$
(9.3)\begin{cases}
\mathrm{Ind}_{g^0}(D) = (m_0+m_1+\cdots\cdots+m_{p-1}) - (n_0+n_1+\cdots+n_{p-1}) = B_0 = 5 \\
\mathrm{Ind}_{g^1}(D) = (m_0+m_1 e^{\frac{2\pi i}{p}}+\cdots+m_{p-1}e^{\frac{2\pi i}{p}(p-1)}) - (n_0+n_1 e^{\frac{2\pi i}{p}}+\cdots+n_{p-1}e^{\frac{2\pi i}{p}(p-1)}) = B_1 \\
\mathrm{Ind}_{g^2}(D) = (m_0+m_1 e^{\frac{2\pi i}{p}2}+\cdots+m_{p-1}e^{\frac{2\pi i}{p}(p-2)}) - (n_0+n_1 e^{\frac{2\pi i}{p}2}+\cdots+n_{p-1}e^{\frac{2\pi i}{p}(p-2)}) = B_2 \\
\quad\vdots \\
\mathrm{Ind}_{g^{p-1}}(D) = (m_0+m_1 e^{\frac{2\pi i}{p}(p-1)}+\cdots+m_{p-1}e^{\frac{2\pi i}{p}}) - (n_0+n_1 e^{\frac{2\pi i}{p}(p-1)}+\cdots+n_{p-1}e^{\frac{2\pi i}{p}}) = B_{p-1}
\end{cases}
$$

Rearrange these equations to compute $(m_0 - n_0)$, $(m_1 - n_1)$, $\cdots$ and $(m_{p-1} - n_{p-1})$

$$(9.4)\begin{cases}(m_0 - n_0) + (m_1 - n_1) + \cdots + (m_{p-1} - n_{p-1}) = B_0 \\ (m_0 - n_0) + (m_n - n_1)e^{\frac{2\pi i}{p}} + \cdots + (m_{p-1} - n_{p-1})e^{\frac{2\pi i}{p}(p-1)} = B_1 \\ (m_0 - n_0) + (m_1 - n_1)e^{\frac{2\pi i}{p}2} + \cdots + (m_{p-1} - n_{p-1})e^{\frac{2\pi i}{p}(p-2)} = B_2 \\ \vdots \\ (m_0 - n_0) + (m_1 - n_1)e^{\frac{2\pi i}{p}(p-1)} + \cdots + (m_{p-1} - n_{p-1})e^{\frac{2\pi i}{p}} = B_{p-1}\end{cases}$$

Using the fact $1 + e^{\frac{2\pi i}{p}} + \cdots + e^{\frac{2\pi i}{p}(p-1)} = 0$. Except the first row and the first column, each row and each column are permutations of the group G. For $0 < k < \ell \le \ell \le p-1$,

$$G \equiv \{1, e^{\frac{2\pi i}{p}k}, e^{\frac{2\pi i}{p}2k}, \cdots, e^{\frac{2\pi i}{p}(p-1)k}\} =' \{1, e^{\frac{2\pi i}{p}(\ell,k)}, e^{\frac{2\pi i}{p}2\cdot(\ell-k)}, \cdots, e^{\frac{2\pi i}{p}(p-1)(\ell-k)}\}$$

By easy calculation we get

$$(9.5)\begin{cases}m_0 - b_0 = \frac{1}{p}(B_0 + B_1 + B_2 + \cdots + B_{p-1}) \\ m_1 - n_1 = \frac{1}{p}(B_0 + B_1 e^{\frac{2\pi i}{p}(p-1)} + B_2 e^{\frac{2\pi i}{p}(p,2)} + \cdots + B_{p-1}e^{\frac{2\pi i}{p}}) \\ \vdots \\ m_{p-1} - n_{p-1} = \frac{1}{p}(B_0 + B_1 e^{\frac{2\pi i}{p}} + B_2 e^{\frac{2\pi i}{p}2} + \cdots + B_{p-1}e^{\frac{2\pi i}{p}(p-1)})\end{cases}$$

REMARK: (i) For a prime number p, $g = e^{\frac{2\pi i}{p}} \in G$, the fixed point set on M, $F = M^G = M^g = M^{g^2} = \cdots = M^{g^{p-1}} = \{P_i\}_{i=1}^{k_1} \cup \{T^{\lambda_i}\}_{i=1}^{k_2}$

(ii) the topological index $B_0 = 5, B_1, \cdots, B_{p-1}$ is determined by the formula (9.2), and the virtual representatoon dimensions $m_i - n_i$ $(i = 0, \cdots, p-1)$ is determined by $B_0, \cdots, B_{p-1}$ and the G-action on them as (9.5).

(iii) For reducible self-dual connections we replace $m_0 + 1$ instead of m_0

(iv) $m_0 - n_0 = \frac{1}{p}(B_0 + B_1 + \cdots + B_{p-1})$ is the dimension of fixed point component containing ∇.

Next we would like to consider a Z_p action on reducible self-dual connections. Under our usual assumption on the bundle $E \to M$. We consider the space $H^2(R)$ of real valued harmonic 2-forms on M. Harmonic 2-forms means $d\phi = 0 = \delta\phi$ where $\delta = -*d*$. Since $\Delta * = \Delta *$, $H^2(R) = H^2_+ \oplus H^2_-$. Every harmonic 2-form is self-dual since the intersection form is positive definite. Also self-dual connections are harmonic. For each reducible connection $\nabla = \nabla_1 \oplus \bar{\nabla}_1$ on E, the assignment $\nabla \mapsto \pm\Omega^{\nabla_1}$ gives a one-to-one correspondence between gauge-equivalence classes of reducible connections on E and pairs of closed 2-forms $\pm\Omega^{\nabla_1}$ with $\int_M \Omega^{\nabla_1} \wedge \Omega^{\nabla_1} = 1$, where $\Omega^{\nabla_1} = \frac{1}{2\pi i} R^{\nabla_1}$. Since M is simply connected, the de Rham classes Ω^{∇_1} are uniquely determined by the integral classes in $H^2(M:Z)$. Also every integral class u with $u \cdot u = 1$ comes from a reducible self-dual connection. Our manifold $M^4 \approx Cp^2 \# \cdots \# Cp^2$ (n-copies).

Let $\{\pm b_1, \cdots \pm b_n\}$ be a basis with $b_i b_j = \delta_{ij}$ in $H^2(M:Z)$. For any $g \in Z_p$, g is a diffeomorphism on M. So $\{\pm g^* b_1, \cdots, \pm g^* b_n\}$ is also a basis in $H^2(M:Z)$. In the moduli space $\mathcal{M}$ there are reducible self-dual connections $\{\nabla_1, \cdots \nabla_n\}$ which corresponds $b_i = \frac{1}{2\pi i} R^{\nabla_i}$ for $i = 1, \cdots, n$. Moreover $g^* b_i = g^* \frac{1}{2\pi i} R^{\nabla_i} = \frac{1}{2\pi i} gR^{\nabla_i} g^{-1} = \frac{1}{2\pi i} R^{g(\nabla_i)}$ which is the curvature from corresponding the connection $g(\nabla_i)$. Since g is an isometry on M, $R^{g(\nabla_i)}$ is harmonic and the integral class $g^* b_i \cdot g^* b_i = g^*(b_i \cdot b_i) = g^*(I) = 1$ where $b_i \cdot b_i$ is a generator in $H^4(M:Z)$. By definition this is the orientation class. Thus $g(\nabla_i)$ is a reducible self-dual connection.

THEOREM 9.6. Z_p-*action on* $E \to M$ *induces a action on the set of reducible connections in the moduli space* $\mathcal{M}$ *of the self-dual gauge equivalent connections. Moreover let* $\check{\mathcal{M}}$ = *the set of reducible connections in* $\mathcal{M}$ *and by setting* $\nabla \longmapsto \frac{1}{2\pi i} R^{\nabla_1}$ *where* $\nabla = \nabla_1 + \bar{\nabla}_1$, *the diagram*

$$\begin{array}{ccc} H^2(M:Z) & \xrightarrow{\ g\ } & H^2(M,Z) \\ \uparrow \Omega & & \uparrow \Omega \\ \check{\mathcal{M}} & \xrightarrow{\ g\ } & \check{\mathcal{M}} \end{array}$$

commutes.

REMARK. (i) Suppose that ∇ is a reducible self-dual connection. Then the isotropy group of ∇, $\Gamma^{\nabla} = \{g \in \mathcal{G} \mid g\nabla g^{-1} = \nabla\} = S^1$. For any $h \in Z_p$, $\Gamma^{h(\nabla)} = \{hgh^{-1} \in \mathcal{G} \mid g \in \Gamma^{\nabla}\} = S^1$ because $(hgh^{-1}) \cdot h(\nabla) = hgh^{-1} \cdot h\nabla h^{-1} \cdot hg^{-1}h^{-1} = hg\nabla g^{-1}h^{-1} = h(\nabla)$. Also $hg_1h^{-1} = hg_2h^{-1} \Rightarrow g_1 = g_2$.

Since Z_p preserves the self-duality, $h(\nabla)$ is also a self-dual reducible connection.

(ii) On the Z_p-action over $E \to M^4$ and $h \in Z_p$. Let $b_1 \in H^2(M:Z)$ with $b_1 \cdot b_1 = 1$, then there is a complex line bundle $L_{b_1} \to M$ with its Euler class b_1. And consider the map

$$\begin{array}{ccc} h^* L_b & \longleftarrow & L_{b_1} \\ \downarrow & & \downarrow \\ M & \xrightarrow{h} & M \end{array} \quad , \quad \begin{array}{rcl} h^*: H^2(M:Z) & \longrightarrow & H^2(M:Z) \\ b_1 & \longmapsto & h^*(b_1) \end{array}$$

The induced bundle $h^*(L_{b_1})$ over M is exactly the bundle corresponding reducible self-dual connections ∇ and $h(\nabla)$

$$\Gamma^\nabla = \{g \in \mathcal{J} | g(\nabla) = \nabla\} = S^1 \subset \mathcal{J}$$

$$\Gamma^{h(\nabla)} = \{ hgh^{-1}\ g \in \Gamma^\nabla | = S^1 \subset \mathcal{J} \text{ and}$$

$g = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \in \Gamma^\nabla$. From the bundle splitting the following diagram commutes and preserves the splittings.

$$\begin{array}{ccc} E = L_{b_1} \oplus \bar{L}_{b_1} & \xrightarrow{h} & L_{h(b_1)} \oplus \bar{L}_{1(b_1)} \\ \Big\downarrow g & & \Big\downarrow hgh^{-1} \\ L_{b_1} \oplus \bar{L}_{b_1} & \xrightarrow{h} & L_h(b_1) \oplus \bar{L}_h(b_1). \end{array}$$

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